

The behavior of the Modified FX-LMS Algorithm with Secondary Path Modelling Errors

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Abstract—In Active Noise Control there has been much research based in the Modified Filtered-X LMS algorithm (MFX-LMS). When the secondary path is perfectly modelled, this algorithm is able to perfectly eliminate its effect. It is also easily adapted to allow the use of fast algorithms such as the RLS, or algorithms with good tracking performance based on the Kalman filter. This paper presents the results of a frequency domain analysis about the behavior of the MFX-LMS in the presence of secondary path modelling errors and a comparison with the FX-LMS algorithm. Namely, it states that for small values of the secondary path delay both algorithms perform the same, but that the step size of the FX-LMS algorithm decreases with increasing delay, while the MFX-LMS algorithm is stable for an arbitrary large value for the secondary path delay, as long as the real part of the ratio of the estimated to the actual path is greater than one half ($\text{Re}\{\hat{S}_z/S_z\} > 1/2$). This means that for the case of no phase errors the estimated amplitude should be greater than half the real one and for the case of no amplitude errors the phase error should be less than 60° . Analytical expressions for the limiting values for the step-size in the presence of modelling errors are given for both algorithms.

Index Terms—Active Noise Control, Secondary Path, FX-LMS, Modelling, RLS, Kalman, LMS.

I. INTRODUCTION

There are a great number of active noise control applications using the FX-LMS algorithm (FX-LMS, fig. 1), however this has several limitations, namely slow convergence due to eigenvalue spread or delay in the secondary path. In fact the step size of the algorithm is limited to $1/(P_x(L + \Delta))$ [1] where Δ is the secondary path delay, L is the filter length and P_x is the filter input power. Also, it is not straight forward to adapt more sophisticated algorithms than LMS such as the RLS [2]

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[3], or algorithms based on the Kalman filter [4] [5] to active noise control using the same approach.

The Modified filtered-X LMS (MFX-LMS) algorithm [6] [7] (fig. 2) algorithm doesn't suffer from maximum step-size reduction with delay in the secondary path. In addition, it removes the secondary path effects in the case of no secondary path modelling errors, allowing easy adaptation of the above mentioned algorithms to active noise control. However, the effects of secondary path modelling errors on the algorithm should be determined. This has been done extensively for the FX-LMS algorithm [8] [9] [10] [11], but there are no results for the MFX-LMS algorithm. Even for the case of the FX-LMS algorithm there are few results for the combined effects of secondary path modelling errors and delay in the secondary path.

This paper studies the behavior of MFX-LMS algorithm with secondary path modelling errors, and makes a comparison with the the FX-LMS algorithm. It is based on the previous published paper [12] but adds some new results, namely the formulas for the bound on the step-size are no longer only for the case of no amplitude errors and no phase errors.

The paper will proceed by studying the stability of the MFX-LMS algorithm for very small step sizes, followed by the determination of stability bounds on the step-size for the case of very large delays, and finally the same analysis is done for the FX-LMS algorithm.

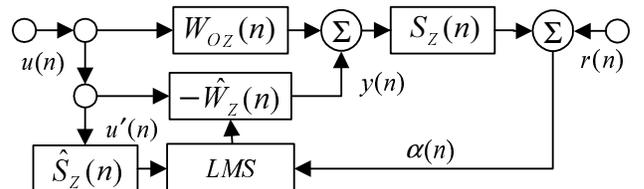


Fig. 1. The Filtered-X LMS algorithms for ANC.

II. ANALYSIS OF THE MODIFIED FILTERED-X LMS

The analysis of the algorithm can be greatly simplified by considering a frequency domain implementation of

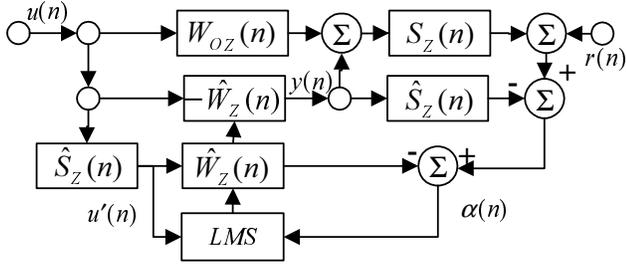


Fig. 2. The Modified Filtered-X LMS algorithm for ANC.

the algorithm [8] [9]. However, a pure frequency domain analysis can only take into account the phase shift in the secondary path, and not the actual delay in the path. Since one of the major differences between the algorithms is the sensitivity to delay in the secondary path, then a pure frequency domain analysis isn't enough. A narrow band analysis will be made instead. More exactly, the secondary path will be defined by a amplitude S , a phase θ_S and a delay d , which corresponds to the complex amplitude S_z , and a group delay d , at a given frequency. This is an accurate model of the secondary path if the anti-noise signal is a narrow band signal at the perturbation frequency [13]. Narrow band analysis results in a big simplification, but it still maintains information about the dynamics of the system to provide some useful insight. In this case, any of the algorithms used in fig. 1 or fig. 2 can be written as:

$$\hat{w}_z(n) = \hat{w}_z(n-1) + \mu u(n-d) \hat{S}_z^* \alpha^*(n) \quad (1)$$

All the signals represent frequency domain values. The dependence on, n , makes this a time frequency analysis or time narrow band analysis. The model of the secondary path is defined by adding a hat at the corresponding symbols for the secondary path, namely, \hat{S} , $\hat{\theta}_S$, \hat{d} , \hat{S}_z . The step size is μ , $\hat{w}_z(n)$ is the complex amplitude of the control filter, $u(n)$ is the complex amplitude of the reference signal, and $\alpha(n)$ is the complex amplitude of the innovations term. For the MFX-LMS algorithm,

$$\alpha(n) = u(n-d)w_{zo}^*S_z^* - u(n-d)\hat{w}_z^*(n-1-d)S_z^* + \quad (2)$$

$$u(n-\hat{d})\hat{w}_z^*(n-1-\hat{d})\hat{S}_z^* + \quad (3)$$

$$-u(n-\hat{d})\hat{S}_z^*\hat{w}_z^*(n-1) + r(n)$$

where $r(n)$ is a measuring noise term, which is uncorrelated with the reference signal. In this paper only the convergence of the mean is going to be studied, which implies that the results should be tested for compliance with experimental results. This is done later in the paper. Replacing $\alpha(n)$ in eq. 1, taking expected values and

letting $R_{\hat{d}\hat{d}} = E[u(n-\hat{d})u(n-\hat{d})]$ and $R_{\hat{d}d} = E[u(n-\hat{d})u(n-d)]$, it is obtained:

$$E[\hat{w}_z(n)] = E[\hat{w}_z(n-1)] + \mu R_{\hat{d}\hat{d}} \hat{S}_z^* S_z w_{zo}(n) - \mu R_{\hat{d}d} \hat{S}_z^* S_z E[\hat{w}_z(n-1-d)] + \mu R_{\hat{d}\hat{d}} \hat{S}_z^* \hat{S}_z E[\hat{w}_z(n-1-\hat{d})] - \mu R_{\hat{d}\hat{d}} \hat{S}_z^* \hat{S}_z E[\hat{w}_z(n-1)] \quad (4)$$

Taking the Z-Transform [14], one gets,

$$E[\hat{w}_z(Z)] = \frac{\mu R_{\hat{d}\hat{d}} \hat{S}_z^* S_z w_{zo}(Z)}{Z-1 + \mu |\hat{S}_z|^2 (R_{\hat{d}\hat{d}}(1-Z^{-\hat{d}}) + R_{\hat{d}d} S_z / \hat{S}_z Z^{-d})} \quad (5)$$

The poles of the system are the values of Z for which the denominator of this expression is equal to zero. Using this equation one can express μ as a function of Z , $\mu = \Gamma(Z)$, where Z is a pole of the system. This means that if a pole of convergence of the adaptive filter is known, then it is possible to calculate the step size used by the algorithm. However, in general this function has no inverse, since there are several modes of convergence, or poles, for a given step size. Nonetheless, for the case of very small step sizes, the convergence is dominated by a single pole, near $Z = 1$, and the inverse exists. Using the rule for the inverse of the implicit function, that is,

$$F(\mu, Z) = 0 \Rightarrow \frac{\partial Z}{\partial \mu} = -\frac{F^1(\mu, Z)}{F^2(\mu, Z)} \quad (6)$$

one can obtain a linear approximation for very small step sizes. In this equation $F^1(\mu, Z)$ represents the derivative in order of the first argument and $F^2(\mu, Z)$ is the derivative in order of the second argument of the function. Since, once more, the algorithm is dominated by the pole at $Z = 1$,

$$Z(\mu) \approx Z(0) - \mu \frac{F^1(0, 1)}{F^2(0, 1)} \approx 1 - \mu e^{-i(\theta_S - \theta_{\hat{S}})} R_{\hat{d}\hat{d}} S \hat{S}. \quad (7)$$

In order to have a stable algorithm for positive values of the step-size, the cross-correlation $R_{\hat{d}\hat{d}}$ should always be positive, which implies that d should equal \hat{d} ; and the phase error absolute value, $\theta_S - \theta_{\hat{S}}$, must be less than 90° . This is the same result as the one obtained for the FX-LMS algorithm and in agreement with what would be intuitively expected.

Now it remains to determine the maximum values for the step size which assure the stability of the algorithm. That is, the value for the step-size which results in the

first crossing of the unit circle by a pole. Since the step size of the algorithm is a real number, a point Z is pole of the system if $\mu = \Gamma(Z)$, as given by eq. 5, is a real number. If the pole is in the unit circle then, Z , is given by $Z = e^{\theta_Z i}$. The paper now proceeds to determine the values of θ_Z for the poles, and then calculate $\mu = |\Gamma(e^{\theta_Z i})|$ to obtain the limiting values on the step-size. To obtain analytical expressions some simplifications are required, which result in sufficient conditions for stability, but which are not always required. It is possible to write,

$$\mu = \Gamma(e^{\theta_Z i}) = \frac{2i \sin(\theta_Z/2) e^{\theta_Z/2i}}{R_{\hat{d}\hat{d}} |\hat{S}_z|^2 (1 - \delta_S e^{-d\theta_Z i})} \quad (8)$$

with,

$$\delta_S = (\hat{S}_z - S_z) / \hat{S}_z. \quad (9)$$

The absolute value of μ is shown in fig. 3. A lower bound for this is,

$$\frac{2 \sin(\theta_Z/2)}{R_{\hat{d}\hat{d}} |\hat{S}_z|^2 (1 + |\delta_S|)} \quad (10)$$

To determine the values of θ_Z of the poles of the system,

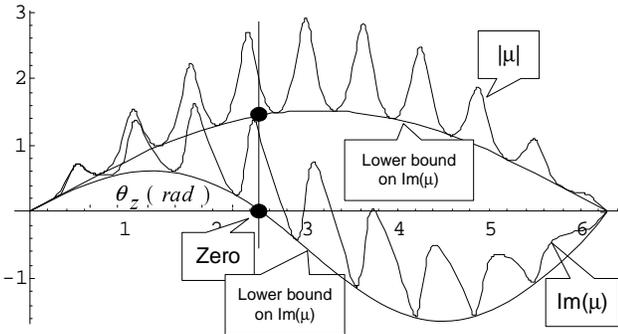


Fig. 3. Plot of $|\mu|$ and $\text{Im}(\mu)$ vs θ_Z for the MFX-LMS algorithm and phase error of 18° . The amplitude follows a $\sin(\theta_Z/2)$ trend.

one must make the imaginary part of μ equal to zero, as in eq. 11. From fig. 3 it can be seen that the smallest values for θ_Z is the one which results in a lower limit for the step-size.

$$\text{Im}\{\mu\} = 0 \Leftrightarrow \text{Re}\{(1 - \delta_S e^{-d\theta_Z i}) e^{-\theta_Z/2i}\} = 0 \quad (11)$$

which is equivalent to,

$$\cos\left(\frac{\theta_Z}{2}\right) = |\delta_S| \cos(\phi), \quad \phi = \left(d + \frac{1}{2}\right) \frac{\theta_Z}{2} \quad (12)$$

If $|\delta_S| > 1$ or $\Re(S_z/\hat{S}_z) < 2$ and if d is large, then this equation has solutions for small values of θ_Z , which limits the step size to very small values, making the algorithm unstable in practice. Otherwise the equation only has solutions for $\theta_Z > \theta_Z'$, with,

$$\cos(\theta_Z'/2) = |\delta_S|. \quad (13)$$

Once again, for large d , small changes in θ_Z , result in large changes in ϕ , namely $\cos(\phi)$ goes from -1 to 1 in only $2\pi/(d + 1/2)$, so the first zero crossing can be taken as $\theta_Z \approx \theta_Z'$. Replacing in equation 10, results in,

$$\mu = \frac{2}{R_{\hat{d}\hat{d}} |\hat{S}_z|^2} \sqrt{\frac{1 - |\delta_S|}{1 + |\delta_S|}}. \quad (14)$$

This equation results in a lower limit for the step size, which guarantees stability. It is plotted in fig. 4, for

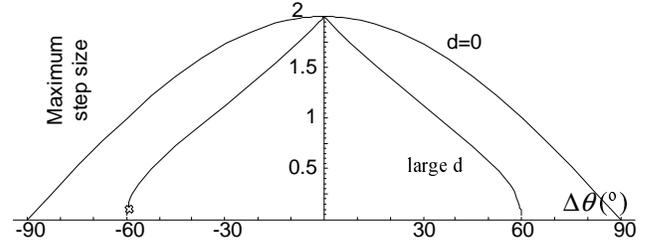


Fig. 4. Value of μ which assures stability versus phase modelling errors for the MFX-LMS algorithm.

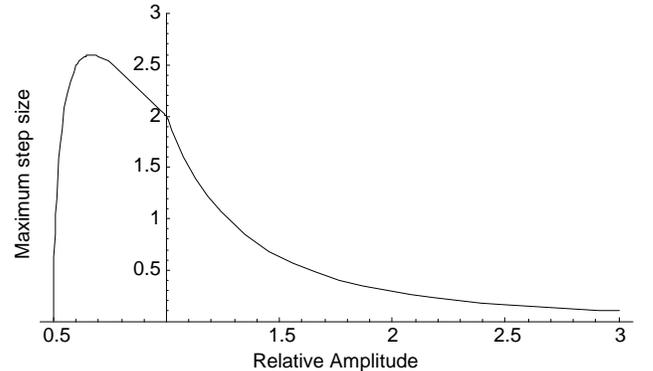


Fig. 5. Value of μ which assures stability versus amplitude modelling errors for the MFX-LMS algorithm.

the case of no amplitude errors and in fig. 5 for no phase errors. It can be seen that it is possible to stabilize the algorithm as long as, $|\delta_S| < 1$ or that, the phase errors are less than 60° for no amplitude errors and the estimated amplitude is more than half of the real one, for no phase errors. Figure 6 shows numerical values for the amplitude of the step size which maintain stability, in function of the phase error. This shows that the limiting values obtained are a bit conservative, but valid.

Figure 7 presents the results of computer simulations. The chart was obtained with a time domain implementation of the MFX-LMS algorithm. The secondary path had a delay of twenty samples, and the reference was a sinusoid synchronously sampled at one fourth of its frequency. The convergence is slower for the case of

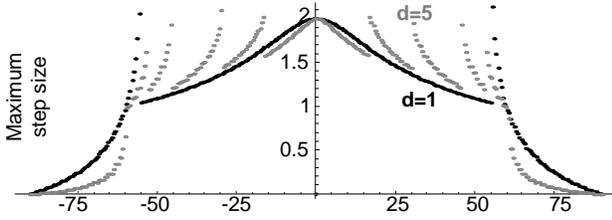


Fig. 6. Numerical values for μ which maintains stability versus phase modelling errors for the MFX-LMS algorithm for a delay of one (dark dots) and five (lite dots).

50° errors. In fact it can be seen that the algorithm has complex poles close to the unit circle, resulting in a spiral convergence path. The maximum values that maintained the stability of the algorithm were almost exactly the ones predicted and the values of the step size which stabilize the algorithm were the ones expected from the previous analysis.

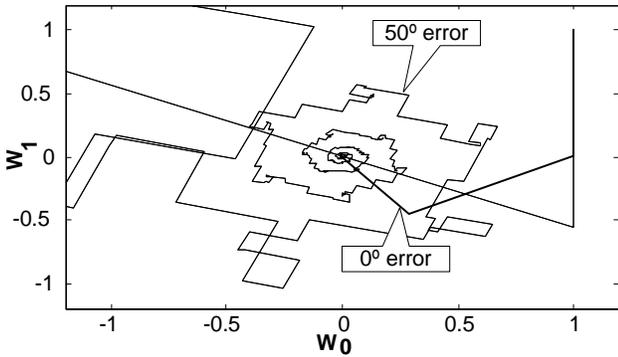


Fig. 7. Simulation results for the convergence of the MFX-LMS algorithm for secondary path modelling phase-errors of 50° and 0° . The normalized step size was one.

III. ANALYSIS OF THE FILTERED-X LMS

There are several works concerning the convergence analysis of the FX-LMS algorithm (fig. 1), so this will only be a brief exposition of the results obtained using a similar approach as the one used for the FX-LMS. The study will only concern convergence of the mean. Replacing $\alpha^*(n)$ for the FX-LMS algorithm in equation 1, and taking expected values, results in,

$$E[\hat{w}_z(n)] = E[\hat{w}_z(n-1)] + \mu R_{\hat{d}\hat{d}} \hat{S}_z^* S_z w_{zo}(n) - \mu R_{\hat{d}\hat{d}} \hat{S}_z^* S_z E[\hat{w}_z(n-1-d)] \quad (45)$$

Taking the Z-transform one can obtain the following condition for the poles of the algorithm,

$$1 - Z^{-1} + \mu R_{\hat{d}\hat{d}} \hat{S}_z^* S_z Z^{-1-d} = 0 \quad (16)$$

Using the rule for the derivative of the implicit function, one obtain the same equation as for the case of the MFX-LMS algorithm, equation 7, showing that both algorithms perform the same way for small values of the step size. The maximum values for the step-size are different however. From equation 16, it is possible to obtain $\mu = \Gamma(Z)$. Making $Z = e^{-\theta_z i}$, as before, the amplitude of μ is given by,

$$\frac{2 \sin(\theta_z/2)}{R_{\hat{d}\hat{d}} S \hat{S}} \quad (17)$$

and the imaginary part is,

$$\cos(\Delta\theta_S - \frac{\theta_z}{2} - \theta_z d) \frac{2 \sin(\theta_z/2)}{R_{\hat{d}\hat{d}} S \hat{S}}, \quad (18)$$

with $\Delta\theta_S = \hat{\theta}_S - \theta_S$. Calculating the zeros and replacing in $|\mu|$ gives for the maximum allowed step size,

$$\min \left(\frac{2 \sin(\frac{\pi/2 \pm \Delta\theta_S}{1+2d})}{R_{\hat{d}\hat{d}} S \hat{S}} \right) \quad (19)$$

This result is plotted in figure 9 for several values of the secondary path delay. For a delay of zero both algorithms perform exactly the same way, namely, the maximum step-size is given by the cosine of the phase error. This is the same result as the one obtained in [9]. For higher delays, the maximum value for the step size is greatly decreased, and the function assumes a different behavior.

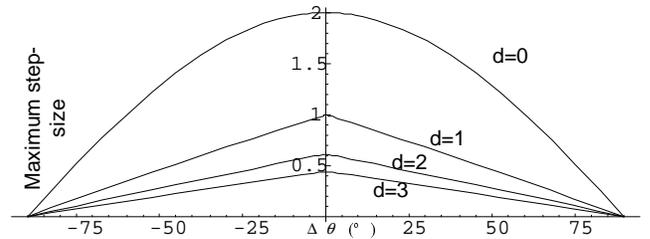


Fig. 8. Value of μ which assures stability versus phase and amplitude modelling errors for the FX-LMS algorithm.

IV. CONCLUSION

Under the simplifying assumption of a narrow band analysis, the MFX-LMS algorithm is stable as long as $\text{Re}\{\hat{S}_z/S_z\} > 1/2$ for arbitrarily large values of the secondary path delay. This means that for no amplitude errors the phase errors should be less than 60° , and for no phase errors the amplitude should be greater than half the real one. The FX-LMS algorithm is stable as long as $\text{Re}\{\hat{S}_z/S_z\} > 0$, but the maximum allowed step-size decreases as the delay in the secondary path increases. For

small values of the step-size both algorithms perform the same, namely they are stable as long as the phase error is less than 90° . Analytical expressions for the bounds on the step-size were derived for both algorithms.

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