Coupled Control of Satellite Platform and Robot Arm for Space Robotics Applications

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Dedicated to my number 1 fan.
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Resumo

Devido a um aumento exponencial de lixo espacial, foi necessário criar missões destinadas à sua remoção. Nesta tese, a fase de captura da missão e.deorbit (a primeira neste campo) é explorada. O satélite que persegue o alvo a ser removido (sendo este também um satélite), utiliza um braço robótico para capturar o alvo. Durante a captura, deve manter a sua posição e atitude, enquanto o braço robótico segue a o ponto do alvo que deverá agarrar.

Para simular a dinâmica e a cinemática do satélite perseguidor, foi implementado um modelo em cadeia aberta. Este modelo denomina-se por model TITOP, que se traduz num modelo de "Duas-Entradas e Duas-Saídas" (Two-Input Two-Output). Esta abordagem permite-nos modelar qualquer corpo com apêndices flexíveis. O satélite perseguidor é, portanto, dividido em vários subsistemas; sendo cada um deles representado por um modelo TITOP. O movimento de líquidos, como combustível, é inserido nos modelos TITOP como um modelo massa-mola, para os subsistemas correspondentes.

O braço robótico produz perturbações no satélite perseguidor e vice-versa; consequentemente, é implementado controlo acoplado. Para o controlador, utilizou-se a abordagem do problema de $H_\infty$ com estrutura fixa. A vantagem aqui é que o a estrutura do controlador é previamente escolhida; por exemplo, nesta tese optou-se por uma estrutura de um controlador PID. De forma semelhante ao método de $H_\infty$ normal, utilizam-se normas de $H_\infty$ para estabelecer certos requisitos. Como desvantagem, esta abordagem podem não convergir.

Finalmente, são analisados estabilidade e desempenho robusto e fornecidas indicações de trabalho futuro.

Palavras-chave: Lixo Espacial, Captura, Braço Robótico, Modelo TITOP, $H_\infty$ com Estrutura Fixa
Abstract

Due to an escalating number of space debris, the need of Active Debris Removal (ADR) mission have become of prime interest. In this thesis, the capture phase of the e.deorbit mission (the first ADR mission) is explored.

The chaser satellite tracks the target satellite, with a robotic arm. The chaser must maintain its pose while the manipulator is tracking the grappling point (GP), defined on the target.

In order to simulate the chaser dynamics and kinematics an open chain model was implemented. This model is called Two-Input Two-Output (TITOP) model. With this approach one is able to model any complex body with flexible appendages. Here, each the chaser system is divided in multiple substructures and each of them is represented by a TITOP model. Sloshing is also considered by introducing a spring-mass-damper system in the TITOP model, for the substructures with sloshing.

The manipulator produces disturbances on the satellite and vice versa, therefore coupled control is considered. During control design, a fixed structured $H_\infty$ control approach was selected. The advantage of using this approach is that we choose the structure of the controller, e.g. in this thesis a PID like structure was implemented, and similarly to the $H_\infty$ control problem, $H_\infty$ norms are used to set requirements.

The main drawback is that it may not converge to a solution.

Finally, robust stability and performance were analysed and directions for future work were provided.

Keywords: e.deorbit Mission, Capture, Robotic Manipulator, TITOP Model, Coupled Control, Fixed Structured $H_\infty$ Control
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Nomenclature

Acronyms
ADCS  Attitude Determination and Control System.
ADR   Active Debris Removal.
AOCS  Attitude and Orbital Control System.
CGFF  Center
CoM   Center of Mass.
DCM   Direction Cosine Matrix.
DH    Denavit Hartenberg.
DoF   Degree of Freedom.
EE    End Effector.
GJM   Generalized Jacobian Matrix.
GP    Grappling Point.
IKP   Inverse Kinematic Problem.
LEO   Low Earth Orbit.
LFT   Linear Fractional Transformation.
PMD   Post Mission Disposal.
RNS   Reaction Null Space.
SVD   Singular Value Decomposition.
TITOP Two-Input Two-Output Port.

Greek symbols
θ     Joint angles.
ϕ     Cost function.
Operation Symbols

⊗ Quaternion multiplication.

Roman symbols

$H_m$ Manipulator’s inertia matrix of some robot.

$H_p$ Platform’s inertia matrix of some robot.

$H_{pm}$ Coupling inertia matrix.

$H$ Inertia matrix of some robot.

$q$ Quaternion.

$J$ Jacobian matrix.

$T$ Homogeneous transformation matrix

Subscripts

$x, y, z$ Cartesian components.

Superscripts

† Pseudo-inverse.

* Conjugate.

T Transpose.
Chapter 1

Introduction

1.1 Motivation

The number of debris in space has become a great concern to operational spacecraft. Cases of uncontrolled re-entry into the atmosphere become an issue of public and property safety. Debris is generated during normal operations by the injection of stages into orbit, release of mission-related objects and retirement of satellites. Further debris is generated due to collision between other debris and other objects in orbit. Collisions trigger further collisions and create more debris leading to a cascade process known as Kessler syndrome [1]. Collisions are not the only concern. Explosion of satellites and launcher upper stages are also a safety hazard due to leftover fuel and partially charged batteries heated up by sunlight [2]. Figure 1.2 shows the evolution of debris from 1960 to 2016, where the cascading effect of debris is evident.

![Figure 1.1: Evolution of debris from 1960 until 2016, from [1].](image)

In Low Earth Orbit (LEO), satellites are exposed to aerodynamic forces from the atmosphere which decelerate debris. This is a major concern for big dimensional debris, such as a non operating satellite, since it could re-entry the atmosphere, leading to possible damages [1–3].

Active Debris Removal (ADR) is more efficient when performed to objects with high mass, high collision
probabilities, and at high altitudes (where there is a larger number of debris due to negligible aerodynamic effect). Different capture mechanisms to ADR methods have been studied. The most promising approaches are listed below [2, 3]:

- **Robotic Arm** - requires a gripper at the end effector (EE), tip of the robotic arm. It can also be used for unmanned satellite servicing, such as satellites refuelling

- **Nets** - if the net is big enough, it has large scalability and low sensitivity to target attitude

- **Tentacles** - ability to capture different targets since the system can easily adapt to a Launch Adapter Ring (LAR) of another satellite with little design review. On one hand, the capture is simplified, on the other hand it requires more accurate rendezvous manoeuvres

- **Harpoons** - insensitive to the target attitude and shape. Does not require close proximity operations

The capture mechanism, considered here, is the Robotic Arm.

Figure 1.2 shows two projected evolutions of the number of objects, larger than 10cm, in LEO. One considers no adherence to satellite debris removal missions (Post Mission Disposal - PMD), while the other one considers an optimistic 90\% adherence to PMD. In 2150, it is expected that the number of fragments considering 90\% PMD is about a third of the number of fragments with no adherence, i.e. 0\% PMD.

Figure 1.2: Projected evolution of the number of objects larger than 10 cm in LEO, with an adherence to Post Mission Disposal (PMD) of 0\% and 90\% [1].

This trend is a great motivation to proceed with ADR. Therefore, ESA is pursuing the first ADR mission ever conducted and it is called e.deorbit. In this mission, a large spacecraft named ENVISAT is to be removed from its orbit. ENVISAT was a clear choice since it is one of the few ESA-owned satellites in a densely populated region between 600-800km altitude [2]. Due to its large size, it has a high risk of collision. The tumbling motion of this spacecraft makes it an even more challenging capture. The tumbling of a non cooperative satellite, such as ENVISAT, requires a synchronization stage preceding the capture itself. Non cooperative satellites do not offer any optical, mechanical, or control aid to the Chaser satellite during stabilization and capture phases.
1.2 Topic Overview

In the context of this dissertation, the capture phase of the *e.deorbit* mission is analysed and in a primary moment the following question must be answered:

*Which control strategies can be applied to solve the capture problem, in an industrial context?*

When designing the controller one must take into account that processors used in space are less computationally efficient than the ones used in common computers, stationed on the ground. This is mainly due to harsh requirements that equipment used in space missions must meet, namely regarding radiation. Hence, simple PD-like controllers are commonly used to reduce computational burden [4, 5]. Space manipulators can be divided into two categories: free-flying robots and free-floating robots [6]. The former may use devices such as reaction wheels and jet thrusters to correct the spacecraft pose. These can be operated simultaneously or intermittently to the manipulator motion, and are controlled by the Attitude Determination and Control System (ADCS) [6, 7]. The desired pose of the end effector (EE) is achieved by controlling joint motors using a manipulator control system. Due to dynamic coupling, the motion of the EE affects the motion of the satellite and vice-versa. In order to avoid undesirable disturbances on the manipulator motion, another approach is to turn off the ADCS; these systems are called free-floating robots [7]. However, in this mission, the pose of the satellite is required to remain stable. Thus, the controller can be divided into two controllers: one for the satellite and another for the manipulator. Therefore, free-flying mode is considered.

During the capture phase, the EE must follow some trajectory given in Cartesian coordinates. This trajectory is then mapped into joint coordinates using the Jacobian matrix $J$. This problem is called the Inverse Kinematic Problem (IKP) [8–10]. The Generalized Jacobian Matrix (GJM) can also be introduced in the guidance. This matrix accounts for the coupling between the satellite and the manipulator. Hence, the EE motion is not perturbed by the satellite motion. More details about GJM are presented in chapter 4 and in [6–8].

A manipulator is kinematically redundant if the number of Degrees of Freedom (DoFs) is higher than the number of task space coordinates [11]. In the IKP with redundant manipulators, we can solve the main task while satisfying some constraints. This is possible by taking advantage of the null space of the Jacobian - joint rates space that do not generate velocity at the EE [12]. In [13], a Closed-Loop Inverse Kinematics (CLIK) algorithm was proposed, which includes feedback for the EE position and orientation. In addition, the null space of the Jacobian is used for joint limit avoidance. Other examples of constraints are velocity limit avoidance, peak torque avoidance, obstacle avoidance, mathematical singularity avoidance, and energy minimization [11–13].

Dynamic control of space robots such as the chaser satellite can be divided into two levels. The first, which was discussed until now, is concerned with coordinate transformations. The second level requires computation of the required force and torque control inputs. In [9], an adaptive control is used for the manipulator controller. This controller is valid for non linear and time-varying plants. In [14], a Linear Quadratic Regulator (LQR) and a $H_{\infty}$ controllers were implemented for a free-floating space manipulator.
with two arms. It was concluded that, in the presence of disturbances and parametric uncertainties, only the $H_\infty$ controller had satisfactory performance. In [7], strategies of control in the presence of angular momentum, for a free-floating space manipulator, are presented. Point-to-point and tracking control in joint space are explored using a PD controller. Control in Cartesian space is also presented using GJM in the IKP, and PD controllers for point-to-point and tracking control.

The fixed structure $H_\infty$ control problem is explored in [4]. As the name suggests, a fixed structure controller is used. Requirements are then defined using the $H_\infty$ norm, which can constrain stability and performance robustness. With this approach, it is possible to obtain a robust controller, and with low computational burden - since the control structure is previously defined. The drawback is that the control problem might not converge. In [4], a PD-like structure was used, which resulted in a good compromise between implementation simplicity and tracking performance.

In this dissertation, a Multi-Input Multi-Output (MIMO) control problem is solved Figure 1.3, summarizes the control system considered in this thesis. The proposed controller is based on a fixed structure $H_\infty$ control problem and a PID-like structure. The result must be robust to the uncertainties of the system. In the control system set-up, an open chain model is used for the plant, where the system is decomposed in different substructures. This model is called Two Input Two Output (TITOP) model and it is introduced in [15]. Since it is based on an open chain model concept, each body corresponds to a node and each edge to a joint between two bodies. With this approach, one is able to model any complex body, regardless of having complex substructures such as robotic arms or flexible solar arrays. The TITOP model can be adapted to include sloshing as a spring-mass-damper system.

![Figure 1.3: Simplified representation of the control architecture.](image)

### 1.3 Objectives and Contributions

The main objective of this dissertation is to design a coupled control system, which keeps the same relative pose of the satellite while the Robotic Arm tracks a point - called Grappling Point (GP). This point is on the target satellite. Since the target is uncooperative and tumbling with an angular rate of 2.5 rad/s, there is an additional challenge to this project: the GP (which is also a reference to the control system) is time variant in the chaser body frame, unless it rotates with the same angular acceleration as
the target.

It should be noted that coupling is considered in the control system by including the effects of joints motion on the satellite, and vice-versa. The control system should also achieve requirements, such as have good performance, and be robust to uncertainties and disturbances of the system.

The contributions of this dissertation are listed below:

- Implementation of an open chain model, the TITOP model, to represent the dynamics and kinematics of the chaser + manipulator system. The TITOP model also introduces sloshing of the fuel and oxidizer tanks as a mass-spring-damper model
- Evaluation of different approaches to the IKP and implementation of singularity avoidance
- Linearisation of the system by implementing the inertia matrix $H$ in the control system
- Compare results between coupled control and independent control, and assess their applicability and limitations
- Develop a fixed structure $H_{\infty}$ control system for the capture phase
- Assessment of the impact of design parameters
- Evaluate the controller performance and robustness

### 1.4 Thesis Outline

The structure of the thesis is as follows:

- Chapter 2 - provides more context regarding the *e.deorbit* and its phases, and it presents physical characteristics of the chaser and target satellites
- Chapter 3 - starts by giving a small background regarding rotating reference frames. The TITOP model follows next. Firstly, all the equations and assumptions behind this model are presented. Next, the implementation of the model in the context of the *e.deorbit* mission is presented. Finally, results corresponding to the validation of the TITOP model are presented, at the end of the chapter
- Chapter 4 - presents the IKP and introduces the Jacobian matrix corresponding to the manipulator. Then, the inertia matrix $H$ of the system is defined. These elements are important for the EE trajectory generation and for the linearisation of the control system
- Chapter 5 - provides a background to control design and presents approaches to obtain some controller, which achieves the objectives, earlier defined. Implementation difficulties are also discussed.
- Chapter 6 - presents results for a nominal and robust solutions
- Chapter 7 - the last chapter contains conclusions and reviews of the project as a whole. Recommendations of future work are also provided in this chapter
Chapter 2

Mission Description

The *e.deorbit* mission is a complex mission, whose goal is to capture a tumbling target and remove it from its orbit. Figure 2.1 summarizes its phases and the respective expected duration.

Launcher lift off and injection in the orbit correspond to the **Launch & Early Observations** phase. Next, **Orbit Transfer & Planning** follows with the transfer of the chaser satellite from launch orbit into Envisat orbit. As the name suggests, in the **Rendezvous** phase, the chaser performs a rendezvous with the target from the Entry Gate - 8 km behind the target - to the Parking Hold Point - 100 m behind the target.

In the **Target Characterization** phase, the Chaser evaluates the structural integrity and attitude dynamics of the Envisat, e.g. tumbling rate and tumbling axis. During **Target Synchronization**, the chaser synchronizes its motion with the target and during **Target Capture** the manipulator tracks the grappling point (GP). It should be noted that the trajectory, which the chaser must follow, is determined by the target angular motion [16], previously observed in Target Characterization. The Chaser + Envisat system is called Stack. Upon confirmation of successful capture, rigidization of the manipulator is performed.

In **Target Stabilization**, the chaser de-tumbles the target, using jet thrusters. Here, joint torques must be closely monitored in order to avoid violations of joint limits. In the **Coupled Flight** phase, the chaser and the Envisat are already coupled and stabilized. Thus, this phase is only concerned with optimization of battery charging. The **Target Fixation** phase ensures that the thrust vector is well aligned with the Stack Center of Mass (CoM). **Stack Orbit Transfer** is responsible for transferring the Stack to the disposal orbit. Finally, in the **Disposal** phase, burn attitude is acquired and executed. Splash down of objects surviving re-entry in the South Pacific Ocean is foreseen [16, 17].

![Figure 2.1: e.deorbit mission phases.](image-url)
2.1  e.deorbit System Overview

In this section, the chaser system is first introduced; its physical characteristics, payload, and Guidance, Navigation and Control (GNC) architecture are presented. Next, an introduction to the Envisat physical characteristics is provided. Figure 2.2 shows a view of both the chaser and the Envisat, during the Capture phase.

![Simulation of the Chaser (in silver) capturing the Envisat (in gold) using a robotic arm](image)

Figure 2.2: Simulation of the Chaser (in silver) capturing the Envisat (in gold) using a robotic arm [16].

### 2.1.1 Chaser

The chaser is a much smaller satellite than its target, Envisat, as noticeable from Figure 2.2. The chaser physical characteristics are summarized in Table 2.1 and Figure 2.3 shows a side view of this satellite, with the robotic arm collected. The robotic arm as an approximate length of 1.25 m, a width of 0.1 m, and a thickness of 0.001 m [18].

An useful frame definition, during this project, is the chaser body frame $\mathcal{F}_C$. This reference frame is represented in Figure 2.3 and it is defined as [18]:

- Origin at the center of the launcher separation plane
- $x$ axis - along the longest axis towards the solar array
- $z$ axis - coincident with launcher longitudinal axis
- $y$ axis - completes the right handed system

The Chaser CoM, in $\mathcal{F}_C$, corresponds to point $[-0.03 \ -0.01 \ 1.13]$ m.

Furthermore, the chaser is equipped with Attitude and Orbit Control System (AOCS) sensors, thrusters for attitude control, robotic payload, etc. Figure 2.4 represents the chaser architecture. Its platform functions correspond to: BUS GNC, Rendezvous GNC, Coupled Control, and Fault Detection, Isolation and Recovery (FDIR) [16]. The BUS GNC is responsible for attitude and control functions outside Rendezvous, Capture and Stabilization. The Rendezvous GNC function takes over during those mission phases. During the Rendezvous phase, collision avoidance manoeuvre is performed; since it depends on the current relative pose and velocity w.r.t. the target, it is calculated in each GNC cycle. Robot arm control functions are also included in the Rendezvous GNC section. Here, joint positions...
Figure 2.3: Chaser side view. The manipulator on top is collected [19].

### Wet Mass at Deorbiting

<table>
<thead>
<tr>
<th></th>
<th>kg</th>
</tr>
</thead>
<tbody>
<tr>
<td>(m)</td>
<td>1380</td>
</tr>
</tbody>
</table>

### Moment of Inertia

<table>
<thead>
<tr>
<th>(I_{xx}) [kg/m²]</th>
<th>(I_{yy}) [kg/m²]</th>
<th>(I_{zz}) [kg/m²]</th>
</tr>
</thead>
<tbody>
<tr>
<td>776</td>
<td>840</td>
<td>307</td>
</tr>
</tbody>
</table>

Table 2.1: Chaser characteristics [18].

Figure 2.4: GNC System Architecture.

commands and joint torque commands are calculated and converted to current in an inner joint motor control loop [16].

FDIR performs health monitoring, by monitoring fault detection and isolation, and switches to redundant equipment when necessary. Force and torques are generated by the control function and translated into thruster opening commands by thruster management function. FDIR disables a specific thruster in case of malfunction, and activates a redundant thruster [16].

The Actuators block can be divided into Attitude Control thruster, Orbit Control thrusters and Assist Engines. It consists of 24 thrusters and it allows commanding forces and torques separately. Their nominal thrust level is 22 N. The orbital control system consists of 2 main engines and 4 assist engines with providing 425 N and 225 N, respectively [20]. During Capture phase, Attitude Control configuration is used in order to control Chaser pose.

AOCS sensors comprise 2 Inertial Measurement Units (IMUs), 3 star trackers, 6 sun sensors, and 2 GPS receivers [18]. Rendezvous sensors include narrow and wide angle cameras, and LIDAR to measure relative pose w.r.t. the target [16, 20].

The Robotic payload consists of the robotic arm, camera, and illumination system. The clamping system, which is composed by a clamping mechanism, a camera and illumination system, will be attached to the Envisat Launch Adapter Ring (LAR), by the end of Capture phase.
Table 2.2 provides information regarding bias and noise associated to the star trackers and the LIDAR, used.

<table>
<thead>
<tr>
<th>Sensors Performance</th>
<th>Type</th>
<th>Bias</th>
<th>Noise</th>
</tr>
</thead>
<tbody>
<tr>
<td>Star Tracker</td>
<td>11 arcsec</td>
<td>2 arcsec</td>
<td></td>
</tr>
<tr>
<td>LIDAR (position)</td>
<td>0.005 m</td>
<td>0.01 m</td>
<td></td>
</tr>
<tr>
<td>LIDAR (velocity)</td>
<td>0.002 m/s</td>
<td>0.002 m/s</td>
<td></td>
</tr>
<tr>
<td>LIDAR (attitude)</td>
<td>0.3 deg</td>
<td>0.15 deg</td>
<td></td>
</tr>
<tr>
<td>LIDAR (angular rate)</td>
<td>0.005 deg/s</td>
<td>0.02 deg/s</td>
<td></td>
</tr>
</tbody>
</table>

Table 2.2: Performance of the sensors attached to the chaser[18].

Finally, Ground Stations are used to monitor and communicate with the chaser.

The performance requirements for the Rendezvous GNC, are summarized in Table 2.3.

<table>
<thead>
<tr>
<th>Position Error</th>
<th>Linear Velocity Error</th>
<th>Attitude</th>
<th>Angular Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.05 m</td>
<td>0.01 m/s</td>
<td>2°</td>
<td>0.5 °/s</td>
</tr>
</tbody>
</table>

Table 2.3: Performance requirements for the Rendezvous GNC, which includes the Capture phase. The values presented are relative quantities [18].

2.1.2 Target

Envisat is a much bigger satellite than the chaser, with an inertia tensor $10 \times$ larger than the chaser inertia. The physical characteristics of the target are summarized in Table 2.4 and the top view of this satellite is represented in Figure 2.5, providing some dimensions.

<table>
<thead>
<tr>
<th>Mass</th>
<th>7828</th>
</tr>
</thead>
<tbody>
<tr>
<td>Moment of Inertia</td>
<td>$I_{xx}$ [kg/m$^2$]</td>
</tr>
<tr>
<td></td>
<td>1700</td>
</tr>
</tbody>
</table>

Table 2.4: Envisat characteristics [18].

Envisat orbits an almost circular LEO, with the orbital parameters presented in Table 2.5.

The GP was defined in the Envisat body frame $\mathcal{F}_T$, which can be defined as follows[18]:

- Origin at the center of Launcher Separation Plane
- $x$ axis - along the longest axis towards the solar array
- $z$ parallel to Ka-Band antenna
### Table 2.5: ENVISAT orbit parameters, during *e.deorbit* mission. [18].

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Altitude [km]</td>
<td>757</td>
</tr>
<tr>
<td>Semi major axis [km]</td>
<td>7135</td>
</tr>
<tr>
<td>Inclination [deg]</td>
<td>98.15</td>
</tr>
<tr>
<td>Eccentricity</td>
<td>0.00117</td>
</tr>
<tr>
<td>Argument of Perigee [deg]</td>
<td>90</td>
</tr>
<tr>
<td>Right Ascension of Ascending Node [km]</td>
<td>0</td>
</tr>
<tr>
<td>Mean Anomaly [deg]</td>
<td>0</td>
</tr>
</tbody>
</table>

- **y** axis - parallel to the ASAR antenna, completing the right handed coordinate system

The GP is positioned in the target LAR, and it is located w.r.t. \( F_T \) at point \([3.90 \, 0.71 \, 1.04] \) m. In the same reference frame, tumbling corresponds to \( 2.5 \hat{g}^o/s \)

### 2.2 Environmental Disturbances

This section describes relevant disturbances in the environment of the chaser satellite. The chaser and the Envisat experience the same environmental disturbances. Hence, only relative values of disturbances actuating on the chaser, w.r.t. the target, are considered in the scope of this project.

#### 2.2.1 Gravitational Force

Newton formulated the law of gravity by stating that two bodies attract each other, with a force proportional to the product of their masses and inversely proportional to the square distance between the two bodies. This law is expressed by [21]

\[
F_{g_{12}} = - \frac{G m_1 m_2}{r_{12}^2} \hat{r}_{12},
\]

where \( G \in \mathbb{R} \) is the universal gravitational constant, \( m_1, m_2 \in \mathbb{R} \) are the masses of body 1 and 2, respectively, \( r_{12} \in \mathbb{R} \) is the distance between, and \( \hat{r}_{12} \in \mathbb{R}^3 \) is the unit vector representing the direction from body 1 to body 2. The gravitational constant has the value \( G = 6.670 \times 10^{-8} \) [21].

The gravitational force between the Earth and the chaser/target is obtained with equation (2.1), by setting \( m_1 \) to the mass of the Earth \( m_E = 5.972 \times 10^{24} \text{kg} \) and \( m_2 \) to the mass of the Chaser/target.

In the context of the *e.deorbit* mission, only the difference between \( F_{g_{EC}} \) and \( F_{g_{ET}} \) is considered - where the former is the gravitational force between the Earth and the chaser, and the latter between the Earth and the target. This differential has an absolute value of \( \approx 3 \text{ N} \). Gravitational force can be introduced in the system as a disturbance, as an alternative to including this force in the model of the system, which facilitates the implementation of the gravitational force in the whole system, as seen in Chapter 4. This is only possible because the variation of the chaser position does not produce a relevant differential of this force.
2.2.2 Gravity Gradient

If the gravitational field were uniform over a material body, its CoM would become the center of gravity and the gravitational torque would be zero. However, in space the gravitational field is not uniform. Variation in the specific gravitational force lead to a gravitational torque about the center of mass. The gravity gradient torque is given by [22]
\[
T_{g12} = 3 \frac{G m_1}{r_{12}^3} \hat{r}_{12} \times I \cdot \hat{r}_{12},
\]
(2.2)
where \( m_1 \) is the mass of the celestial primary (body 1) and \( I \in \mathbb{R}^{3\times3} \) is the inertia tensor of body 2. Again, only the differential value of the gravity gradient between the Chaser and the Target, w.r.t. the Earth, matter. The absolute value of this differential is approximately \( 3 \times 10^{-7} \text{Nm} \). Therefore, it is neglected in the context of this mission.

2.2.3 Drag Force

The drag force is considered to be among the most predominant among other perturbing forces (disregarding gravitational force) [23]. It exists due to residual atmosphere, whose molecules impact on the spacecraft surface [24]. Solar activity has a major influence on the upper atmosphere and near Earth environment. The main driver of the atmosphere density changes is the varying solar activity: the Sun heats and expands the upper atmosphere which increases atmospheric drag. Long term trends, such as the 11-year solar cycle, and short term trends, e.g. localised effects due to diurnal and seasonal changes, vary the atmosphere density [23].

The aerodynamic drag force is given by the equation [24]
\[
F_d = -\frac{1}{2} \rho C_d |v|^2 A v,
\]
(2.3)
where \( \rho \in \mathbb{R} \) is the air density, \( C_d \in \mathbb{R} \) the drag coefficient, \( v \in \mathbb{R}^3 \) is the relative velocity of the satellite w.r.t. the atmosphere, and \( A \in \mathbb{R} \) the cross sectional satellite area, perpendicular to the relative velocity vector.

For relative manoeuvres, one is interested in the differential drag force between the chaser and the Envisat drag forces. Assuming the same relative velocity, w.r.t. the atmosphere, for both satellites, the absolute value of the differential drag force acting on the chaser is expressed by
\[
\Delta F_{dC} = -\frac{1}{2} C_d \rho |v|^2 \left( A_C - A_T \frac{m_C}{m_T} \right),
\]
(2.4)
where \( A_C \) and \( A_T \) are the cross sectional areas of the chaser and the target, respectively; \( m_C \) is the chaser mass and \( m_T \) the target mass; for typical spacecraft \( C_D = 0.2 \) [24], \( \rho = 3.614 \times 10^{-14} \text{kg/m}^3 \) [25]. For a reference value, one can use the escape velocity in \( |v| \), which for an altitude of 757 km, is \( 3.344 \times 10^5 \). With these values one obtains a drag force of the order \( 10^{-3} \), which can be neglected.
comparing to the gravitational force.
Chapter 3

Relative Dynamics and Kinematics -
The TITOP Model

Spacecrafts are very complex multi-body systems, with flexible and rotating appendages.

The design of some linear control system requires a linear model that represents the dynamics of the system. In particular, the model presented in this chapter takes into account the rotation of the appendages and their flexibility. It is noted that the linearisation is valid if slow movements of the hub and its appendages are considered [26].

In [15], an open chain model, where forces and torques are propagated from element to element, is presented. In this model, each node corresponds to a body and each edge to a joint between two bodies (tree terminology). Thus, each element can be represented as a multi-input multi-output transfer with two channels: direct dynamic model and inverse dynamic model. The first one is at the connection point between some substructure and its parent (previous substructure), where accelerations are inputs and forces and torques are outputs. The second one is at the connection point between the substructure and its child (next substructure); here, forces and torques are inputs and accelerations are outputs. This model is called Two Input Two-Output Port (TITOP) model.

Such a model allows us to build the dynamic model of an entire spacecraft by assembling a TITOP model to each substructure. This model also enables the designer to isolate parameters to be optimized.

3.1 Model Formulation

Hereinafter, all the equations and assumptions behind the TITOP model are presented. Firstly, the transformation equations in a rotating frame will be discussed, and will be used as a background to the TITOP model.
3.1.1 Velocity and Acceleration in Rotating Frames

In [27], the kinematics in rotating frames are explained in more detail. Henceforth, the variables written in upper-case are w.r.t. reference frame $\mathcal{F}_i$ and the ones written in lower-case are w.r.t. frame $\mathcal{F}_s$. In this regard, $\vec{V}_A^i \in \mathbb{R}^3$ stands for the linear velocity vector of some point A, measured in a fixed frame $\mathcal{F}_i$; $\vec{v}_p^s \in \mathbb{R}^3$ corresponds to the linear velocity of the same point, measured in a rotating frame $\mathcal{F}_s$ instead.

Figure 3.1 represents both fixed $\mathcal{F}_i$ and rotating $\mathcal{F}_s$ frames.

Vector $\vec{R}_A \in \mathbb{R}^3$ points from the origin of frame $\mathcal{F}_i$ to point A and it is given by

$$\vec{R}_A = \vec{R}_{O'} + \vec{r},$$

where $\vec{R}_{O'} \in \mathbb{R}^3$ points from the origin of frame $\mathcal{F}_i$ to the origin of frame $\mathcal{F}_s$; $\vec{r} \in \mathbb{R}^3$ is the position vector of point A w.r.t. frame $\mathcal{F}_s$ and it is expressed by

$$\vec{r} = a_1 \vec{s}_1 + a_2 \vec{s}_2,$$

where $a_1, a_2 \in \mathbb{R}$ are the coordinates in axis $s_1$ and $s_2$, respectively. Figure 3.1 shows the vectors mentioned above.

The inertial linear velocity at point A, $\vec{V}_A$, can now be expressed by

$$\vec{V}_A = \frac{d}{dt} \left( \vec{R}_{O'} + \vec{r} \right) = \frac{d}{dt} \vec{R} + \frac{d}{dt} \vec{r} + a_1 \dot{\vec{s}}_1 + a_2 \dot{\vec{s}}_2.$$

Figure 3.2: Representation of the rotation of reference frame $s$
A small rotation around the axis \( s_3 \) (see figure 3.2) yields

\[
\begin{align*}
\dot{s}_1 &= \lim_{\Delta t \to 0} \frac{\Delta s_2}{\Delta t} = \dot{\theta} s_2 \\
\dot{s}_2 &= \lim_{\Delta t \to 0} \frac{\Delta s_1}{\Delta t} = -\dot{\theta} s_1
\end{align*}
\]

(3.1)

The rotation rate around the axis \( s_3, \omega^s \in \mathbb{R}^3 \), is given by \( \omega^s = \omega^s s_3 \). The set of equations in (3.1) can also be written as

\[
\begin{align*}
\dot{s}_1 &= \omega^s \times s_1 \\
\dot{s}_2 &= \omega^s \times s_2
\end{align*}
\]

Therefore, the linear velocity at point A, w.r.t fixed frame \( F_i \), can be written as

\[
\vec{V}_A = \vec{V}_{O'} + \frac{d}{dt} \vec{r} + \omega^s \times \vec{r}.
\]

(3.2)

In order to obtain the inertial linear acceleration at point A, one must derive equation (3.2), which yields

\[
\vec{a}_A = \frac{d}{dt} \vec{V}_A
\]

\[
= \frac{d}{dt} \vec{V}_{O'} + \frac{d}{dt} \left( \frac{d}{dt} \vec{r} + \omega^s \times \vec{r} \right)
\]

\[
= \vec{a}_{O'} + \vec{a}^s + \omega^s \times (\omega^s \times \vec{r}) + \dot{\omega}^s \times \vec{r} + 2\omega^s \times \vec{v}^s,
\]

(3.3)

where \( \vec{a}_{O'} \in \mathbb{R}^3 \) is the acceleration at the origin of frame \( F_s \), in frame \( F_i \); \( \vec{a}^s \in \mathbb{R}^3 \) is the linear acceleration at point A, written in frame \( F_s \); \( \omega^s \times (\omega^s \times \vec{r}) \) is the centrifugal acceleration, \( \dot{\omega}^s \times \vec{r} \) is the angular acceleration of frame \( F_s \) measured in frame \( F_i \), and \( 2\omega^s \times \vec{v}^s \) is the Coriolis acceleration.

The centrifugal acceleration term is due to the rotation of frame \( F_s \) w.r.t. frame \( F_i \).

### 3.1.2 Dynamic & Kinematic Equations

Newton’s Second Law \( \vec{F} = m \vec{a} \) is only valid in an inertial frame. A reference frame centred in some rotating spacecraft CoM is not inertial. Thus, in order to apply Newton’s Second Law to its CoM, one must take into account the components due to the rotation of the frame: centrifugal acceleration, Coriolis acceleration and the angular acceleration of the rotating frame w.r.t an inertial frame. Hence, the inertial acceleration \( \vec{a} \in \mathbb{R}^3 \), must be written in terms of (3.3).

Back to the TITOP model, a representation of some structure is depicted in figure 3.3, with forces and torques being applied on body \( A \). Points P and C correspond to connection points between its parent and child, respectively. Point A represents the substructure CoM. The parent and child of body \( A \) corresponds to bodies \( P \) and \( C \), respectively.

The resulting force \( \vec{F}_{ext} \in \mathbb{R}^3 \) that is being applied on body \( A \), is given by

\[
\vec{F}_{ext} = m_A \vec{a}_A
\]

\[
= \vec{F}_{C/A} + \vec{F}_{P/A},
\]

(3.4)
where \( m_A \in \mathbb{R} \) is the mass of body \( A \) and \( \vec{a}_A \in \mathbb{R}^3 \) is the inertial acceleration of body \( A \), at its center of mass \( A \); \( \vec{F}_{P/A}, \vec{F}_{C/A} \in \mathbb{R}^3 \) are the forces that the parent and child bodies apply on \( A \), respectively.

The total external torques \( \vec{T}_{ext} \in \mathbb{R}^3 \), applied on \( A \), is expressed by
\[
\vec{T}_{ext} = \vec{J}_{A} \dot{\vec{ω}}_A + \vec{ω}_A \times \vec{J}_{A} \vec{ω}_A, \tag{3.5}
\]

where \( \vec{J}_{A} \in \mathbb{R}^{3 \times 3} \) is the inertia matrix of body \( A \), calculated around \( A \), \( \vec{ω}_A \in \mathbb{R}^3 \) is the angular velocity at point \( A \), measured in the inertial frame, and \( \dot{\vec{ω}}_A \) is the time derivative of \( \vec{ω}_A \), i.e. the angular acceleration;

\( \vec{T}_{(\cdot)\langle\cdot,\cdot\rangle} \in \mathbb{R}^3 \) is the torque that body \( (\cdot) \) applies on body \( (\cdot) \), at the connection point \( (\cdot) \), \( \vec{A}P \in \mathbb{R}^3 \) is the position vector from point \( A \) to \( P \), and \( \vec{AC} \) points from \( A \) to \( C \).

The dynamics of body \( A \) can be expressed by equations (3.4) and (3.6).

One of the two inputs corresponding to the inverse dynamic channel is the inertial acceleration of body \( A \) at point \( P \): \( \vec{a}_{AP} \in \mathbb{R}^3 \). Therefore, (3.4) and (3.6) must be reformulated as functions of \( \vec{a}_{AP} \).

First, a reference frame centred in \( P \) and rotating with an angular velocity of \( \vec{ω}_{AP} \in \mathbb{R}^3 \) is defined. Then, we substitute \( \vec{r} \) in equation (3.3) by a position vector from point \( P \) to point \( A \), \( \vec{PA} \).

If body \( A \) is rigid, \( \vec{PA} \) is constant, i.e. \( \vec{v}_A = \dot{\vec{PA}} = 0 \) and \( \vec{a}_A = \ddot{\vec{PA}} = 0 \). Hence, the acceleration at point \( A \) is given by
\[
\vec{a}_A = \vec{a}_{AP} + \dot{\vec{ω}}_A \times \vec{PA} + \vec{ω}_A \times (\vec{ω}_{AP} \times \vec{PA}). \tag{3.7}
\]

Furthermore, if the angular velocity \( \vec{ω}_{AP} \) of the rotating frame is small enough, we can neglect the centrifugal acceleration. Thus, the linear acceleration at point \( A \) is given by
\[
\vec{a}_A = \vec{a}_{AP} + \dot{\vec{ω}}_A \times \vec{PA}. \tag{3.7}
\]

All bodies in the chain model of Figure 3.3 are rigid. Thus, equation \( \dot{\vec{ω}}_P = \dot{\vec{ω}}_A \) holds and yields
\[
\begin{cases}
\vec{F}_{C/A} + \vec{F}_{P/A} = m_A (\vec{a}_{AP} + \dot{\vec{ω}}_P \times \vec{PA}) \\
\vec{T}_{C/A,P} + \vec{T}_{P/A} + \vec{AP} \times \vec{F}_{P/A} + \vec{AC} \times \vec{F}_{C/A} = \vec{J}_{A} \dot{\vec{ω}}_A
\end{cases}. \tag{3.8}
\]

It should be noted that, in (3.5), the last term was neglected.

We know that \( \dot{\vec{ω}}_P = \dot{\vec{ω}}_A \). If we consider that the whole system has the same initial conditions, then \( \vec{ω}_P = \vec{ω}_A \) also holds.

The set of linearised equations (3.8), which expresses the dynamics of body \( A \), can now be implemented in the design of a linear controller. The linearisation effects will be studied in section 3.3.
Similarly to equation (3.7), the acceleration at point C of body \( A \), \( \ddot{a}_C^A \in \mathbb{R}^3 \), is given by

\[
\ddot{a}_C^A = \ddot{a}_P^A - \omega_P^A \times \dot{p}_C^A
\]

Linear and angular accelerations are transferred from body \( A \) to its child body \( C \), using the following equations

\[
\begin{align*}
\ddot{a}_C^A &= R \ddot{a}_P^C \\
\dot{\omega}_C^A &= R \dot{\omega}_P^C
\end{align*}
\]  

(3.9)

where \( \ddot{a}_P^A, \dot{\omega}_P^A \in \mathbb{R}^3 \) are the linear and angular velocities of body \( \cdot \) at point \( \cdot \). From Figure 3.3 we can see that points P and C are coincident in neighbouring bodies. In eq. (3.9), \( R \in \mathbb{R}^{3 \times 3} \) is the Direction Cosine Matrix (DCM) that represents the transformation from frame \( \mathcal{F}_A \) to frame \( \mathcal{F}_C \). Reference frame \( \mathcal{F}_A \) is centred in \( A \) and rotating with an angular rate of \( \dot{\omega}_A \), and the latter is centred in body \( C \) and with an angular rate of \( \dot{\omega}_C \). If both rotating frames are rotating with the same inertial angular velocities, then \( R \) is constant.

The external torque \( \vec{T}_{ext}^P \in \mathbb{R}^3 \) acting on point P of body \( A \) is given by

\[
\vec{T}_{ext}^P = \vec{T}_{ext}^A + \vec{F}_A \times \vec{F}_{ext}.
\]  

(3.10)

Equation (3.10) can also be formulated as

\[
\vec{T}_{ext}^P = \vec{T}_{P/A,P} + \vec{T}_{C/A,C} + \vec{F}_C \times \vec{F}_{A/C}.
\]  

(3.11)

Substituting \( \vec{T}_{ext}^A \) by \( J_A^A \dot{\omega}_A \) and \( \vec{F}_{ext} \) by \( m_A^A (\ddot{a}_P + \dot{\omega}_P \times \vec{F}_A) \), and by equating equations (3.10) and (3.11), we obtain

\[
\vec{T}_{P/A,P} = -\vec{T}_{C/A,C} - \vec{F}_C \times \vec{F}_{A/C} + m_A^A \vec{F}_A \times \ddot{a}_P^A + J_A^A \dot{\omega}_P^A + m_A^A \vec{F}_A \times \dot{\omega}_P^A \times \vec{F}_A.
\]  

(3.12)

Now, we have all the equations necessary to define the direct and indirect channels mentioned in the beginning of this chapter. These equations are given by

\[
\begin{align*}
\ddot{a}_C^A &= \ddot{a}_P^A + \dot{\omega}_P^A \times \dot{p}_C^A \\
\dot{\omega}_C^A &= \dot{\omega}_P^A \\
\vec{F}_{A/P} &= \vec{F}_{C/A} - m_A^A (\ddot{a}_P^A + \dot{\omega}_P^A \times \vec{F}_A) \\
\vec{T}_{A/P,P} &= \vec{T}_{C/A,C} - \vec{C}_P \times \vec{F}_{C/A} + m_A^A \vec{A}_P \times \ddot{a}_P^A - J_A^A \dot{\omega}_P^A + m_A^A \dot{\omega}_P^A \times \vec{A}_P \times \vec{A}_P
\end{align*}
\]  

(3.13)

where \( \vec{F}_{A/P} = -\vec{F}_{P/A} \) and \( \vec{T}_{A/P,P} = -\vec{T}_{P/A,P} \).

The first two equations of (3.13) solve the kinematics between two substructures and the last two solve the dynamics between the same two bodies.

To implement the set of equations (3.13) in a TITOP block, one must write them in a matricidal format,
obtaining
\[
\begin{bmatrix}
\mathbf{a}_C \\
\mathbf{F}_{A/\mathcal{T}, P}
\end{bmatrix}
= M^A_P
\begin{bmatrix}
\mathbf{F}_{e/A, C} \\
\mathbf{a}_P
\end{bmatrix},
\tag{3.14}
\]
where \(\mathbf{a}_C = [\mathbf{\dot{a}}_C^A, \mathbf{\ddot{a}}_C^A]^T \in \mathbb{R}^6\) and \(\mathbf{F}_{A/\mathcal{T}, P} = [\bar{\mathbf{F}}_{A/P}, \bar{T}_{A/\mathcal{T}, P}]^T \in \mathbb{R}^6\), and the same follows for \(\mathbf{a}_P\) and \(\mathbf{F}_{e/A, C}\). Matrix \(M^A_P \in \mathbb{R}^{12 \times 12}\) is given by
\[
M^A_P = \begin{bmatrix}
\mathbf{0}_{6 \times 6} & \tau_{CP} \\
\tau_{CP}^T & -\mathbf{D}_{AP}
\end{bmatrix},
\tag{3.15}
\]
where \(\tau_{CP} \in \mathbb{R}^{6 \times 6}\) is the kinematic model between points C and P and it is expressed by
\[
\tau_{CP} = \begin{bmatrix}
\mathbf{I}_3 \\
\mathbf{0}_{3 \times 3} \\
\mathbf{I}_3
\end{bmatrix} (CP \times).
\tag{3.15}
\]

The dynamic model between point A and point P is represented by \(\mathbf{D}_{AP} \in \mathbb{R}^{6 \times 6}\) and it is given by [15]
\[
\mathbf{D}_{AP} = \begin{bmatrix}
\mathbf{m}^A \mathbf{I}_3 & \mathbf{m}^A (AP \times) \\
-\mathbf{m}^A (AP \times) & \mathbf{J}_A \mathbf{m}^A (AP \times)^2
\end{bmatrix},
\tag{3.16}
\]
where \((AP \times) \in \mathbb{R}^{3 \times 3}\) is a skew symmetric matrix given by
\[
(AP \times) = \begin{bmatrix}
0 & -z & y \\
z & 0 & -x \\
y & x & 0
\end{bmatrix}.
\tag{3.17}
\]

Three by three skew-symmetric matrices can be used to represent cross products as matrix multiplication: \((AP \times) r = \overrightarrow{AP} \times r\), where \(r \in \mathbb{R}^3\). Another important property of these types of matrices is that their symmetric matrices satisfies: \(- (AP \times) = (AP \times)^T\).

Using equation (3.14), we can build the TITOP block represented in Figure 3.4.

![Figure 3.4: Representation of a TITOP block.](image)

\(\mathbf{a}_P = [\mathbf{\dot{a}}_P^A, \mathbf{\ddot{a}}_P^A]^T\) and \(\mathbf{F}_{A/\mathcal{T}} = [\bar{\mathbf{F}}_{A/P}, \bar{T}_{A/\mathcal{T}, P}]^T\) are the input and output of the inverse dynamic channel, respectively. The input and output of the direct dynamic channel are, respectively, represented by: \(\mathbf{F}_{e/A} = [\bar{\mathbf{F}}_{e/A}, \bar{T}_{e/A,C}]^T\) and \(\mathbf{a}_C = [\mathbf{\dot{a}}_C^A, \mathbf{\ddot{a}}_C^A]^T\).

It should be noted that external commands are applied at the joints of the robotic arm, in order to control the position of its End Effector (EE). These external commands are provided as torques, at each joint. We shall consider that joint \(i\) is positioned at point P of segment \(i\) (where \(i \in [1, n]\) and \(n\) is the number
of joints) - see Figure 3.3. The external torque \( \vec{C}_m \in \mathbb{R}^3 \) at the joint of some body \( \mathcal{A} \) can be written as

\[
\vec{C}_m = -\hat{r}_c \cdot \vec{T}_{A/P,P},
\]

where \( \hat{r}_c \in \mathbb{R}^3 \) is the unit vector that represents the axis of rotation of the revolute joint (1 degree of freedom), w.r.t. the body frame corresponding to \( \mathcal{A} \) [15, 26].

As a consequence of this extra torque, the angular acceleration also changes. Thus, after applying the command \( \vec{C}_m \) the angular acceleration at point C of the same body becomes

\[
\dot{\vec{\omega}}_C^A = \dot{\vec{\omega}}_P^A + \ddot{\theta},
\]

where \( \dot{\vec{\omega}}_P^A \in \mathbb{R}^3 \) is the angular acceleration at point P of segment \( \mathcal{A} \) before the command. Also, note that

\[
\dot{\vec{\omega}}_P^A = R^T \dot{\vec{\omega}}_C^A,
\]

where \( R \) is the DCM from \( F_A \) to \( F_P \).

Given that the external torque \( C_m \) is applied at point P, this is the center of rotation when considering this torque alone.

If we choose a frame rotating with an angular rate of \( \dot{\vec{\omega}}_P^A + \ddot{\theta} \), then every point \( j \) of body \( \mathcal{A} \) is fixed w.r.t. the rotating frame, i.e. \( \vec{v}^A_j = 0 \) and \( \vec{a}^A_j = 0 \). Thus, the linear acceleration is expressed by

\[
\vec{a}^A_j = \vec{a}^A_P + (\dot{\vec{\omega}}_P^A + \ddot{\theta}) \times (\vec{a}^A_P + \ddot{\theta}) \times \vec{r}.
\]

One should take into account that when transferring the linear and angular accelerations from some body to its rotating child node, the transformation matrix \( R \) is now not constant - \( \theta \) is variable. Although the TITOP model is based on linear dynamics and kinematics equations, this model cannot be considered as linear due to the time variant DCMs corresponding to the manipulator segments.

Neglecting the squared components of the angular rates, the set of equations for the inverse and the direct channels are now expressed by:

\[
\begin{align*}
\vec{a}^A &= \vec{a}^A_P + (\dot{\vec{\omega}}_P^A + \ddot{\theta}) \times \vec{P}C \\
\dot{\vec{\omega}}_C^A &= \dot{\vec{\omega}}_P^A + \ddot{\theta} \\
\vec{F}_{A/P} &= \vec{F}_{C/A} - m^A \left[ \vec{a}^A_P + (\dot{\vec{\omega}}_P^A + \ddot{\theta}) \times \vec{A}P \right] \\
\vec{T}_{A/P,P} &= \vec{T}_{C/A,C} - \vec{C}P \times \vec{F}_{C/A} - m^A \vec{A}P \times \vec{a}^A_P + J_A \left( \dot{\vec{\omega}}_P^A + \ddot{\theta} \right) - m^A \vec{A}P \times \left[ \vec{A}P \times (\dot{\vec{\omega}}_P^A + \ddot{\theta}) \right] \\
C_m &= -\hat{r}_c \cdot \vec{T}_{A/P,P}
\end{align*}
\]

Writing equations (3.19) in a matricidal format, we obtain
\[
\begin{bmatrix}
a_C \\
F_{A/P} \\
C_m \\
\dot{\theta}
\end{bmatrix} = M_{P}^{A}
\begin{bmatrix}
a_p \\
\dot{a}_P \\
\dot{\theta}
\end{bmatrix},
\] (3.20)

where \(C_m, \dot{\theta} \in \mathbb{R}\) and \(M_{P}^{A} \in \mathbb{R}^{13 \times 13}\) is equivalent to

\[
M_{P}^{A} = \begin{bmatrix}
0_{6 \times 6} & \tau_{CP}^T & (CP \times) r_e \\
\tau_{C}^T & -D_{AP} & (CP \times) r_e \\
0_{1 \times 6} & m^2 r_e^T (AP \times) r_e - m^2 r_e^T (AP \times)^2 r_e & \tau_{C}^T J_{A} r_e - m^2 r_e^T (AP \times)^2 r_e
\end{bmatrix},
\]

where \(r_e \in \mathbb{R}^3\) is an unit vector that represents a revolute joint axis of rotation.

The TITOP block, representing the dynamics and kinematics expressed by equation (3.20), is represented in figure 3.5.

Figure 3.5: Representation of an augmented TITOP block \(a_P = [\dot{a}_P, \dot{\omega}_P]^T\) and \(F_{A/P} = [F_{A/P}, \dot{T}_{A/P}]^T\) are the input and output of the inverse dynamic channel, respectively. The inputs are \(F_{C/A} = [\dot{F}_{C/A}, \dot{T}_{C/A}]^T\) and the torque at the joint \(C_m\); the outputs of this channel are \(a_C = [\dot{a}_C, \dot{\omega}_C]^T\) and the angular acceleration \(\dot{\theta}\) at the joint.

**Flexible appendages**

When a large structural component is cantilevered within another part of the spacecraft, it might be necessary to include it as flexible appendages. Examples can be solar arrays and antennas.

The dynamics of the main body is different from the dynamics of a flexible body dynamics. Each flexible appendage is connected to the main body by a cantilever joint and its dynamics yield [28]

\[
F_{P/A,P} = D_{AP} a_P + L_P \eta,
\] (3.21)

where \(L_P \in \mathbb{R}^{6 \times N}\) is called the Modal Participation Factor Matrix and \(N \in \mathbb{R}\) is the number of flexible modes; \(\eta \in \mathbb{R}^N\) is the Model Coordinates Vector of flexible modes and yields

\[
\ddot{\eta} = \text{diag}(2\xi_i \omega_i) \omega_P + \text{diag}(w_i^2) \eta = -L_P^T a_P,
\] (3.22)

where \(\omega_i \in \mathbb{R}\) corresponds to a flexible mode and \(\xi_i \in \mathbb{R}\) is a standard damping ratio, with \(i \in [1, N]\). The Stiffness Model is given by \(K = \text{diag}(\omega_i^2) \in \mathbb{R}^{N \times N}\) and the Damping Matrix is \(D = \text{diag}(2\xi_i \omega_i)\).
The acceleration vector at the connection point C \([\vec{a}_C, \dot{\omega}_C]^T\) is given by

\[
\begin{bmatrix}
\vec{a}_C \\
\dot{\omega}_C
\end{bmatrix} =
\begin{bmatrix}
\delta \vec{a}_C \\
\delta \dot{\omega}_C
\end{bmatrix} + \tau_{CP}
\begin{bmatrix}
\vec{a}_P \\
\dot{\omega}_P
\end{bmatrix},
\]

(3.23)

where \(\tau_{CP}\) is the kinematic model between points C and P and given by equation (3.15), \([\delta \vec{a}_C, \delta \dot{\omega}_C]^T\) is the acceleration contribution from the flexible modes; \([\vec{a}_P, \dot{\omega}_P]^T\) corresponds to the already known rigid component of the acceleration.

Equations (3.21), (3.22), (3.23) and \(F_{A/P} = -F_{A/P}\) yield the following state space for a hybrid cantilever model [28]

\[
\begin{bmatrix}
\dot{\eta} \\
\ddot{\eta}
\end{bmatrix} =
\begin{bmatrix}
0_{N \times N} & I_N \\
-K_{N \times N} & -D_{N \times N}
\end{bmatrix}
\begin{bmatrix}
\eta \\
\dot{\eta}
\end{bmatrix} +
\begin{bmatrix}
0_{N \times 6} \\
L_P
\end{bmatrix}
\begin{bmatrix}
\vec{a}_P \\
\dot{\omega}_P
\end{bmatrix}
\]

\[
\begin{bmatrix}
\vec{F}_{A/P} \\
\bar{T}_{A/P, P}
\end{bmatrix} =
\begin{bmatrix}
L_P^T K & L_P^T D \\
I_P & L_P^T L_P
\end{bmatrix}
\begin{bmatrix}
\eta \\
\dot{\eta}
\end{bmatrix} +
\begin{bmatrix}
-D_{AP} & L_P^T L_P
\end{bmatrix}
\begin{bmatrix}
\vec{a}_P \\
\dot{\omega}_P
\end{bmatrix}.
\]

It should be noted that \(D_{AP}\) corresponds to the dynamic model of the body, given by (3.16); and \(D\) corresponds to the Damping Matrix mentioned above.

### 3.2 Model Implementation

Here, all the steps to implement the TITOP model are explained. First, one ought to divide the entire structure of the spacecraft into smaller substructures.

The Chaser was divided into the following substructures:

- Hub - main body, where the CoM of the spacecraft is located;
- Fuel tank - a flexible appendage (sloshing effect);
- Oxidizer tank - a flexible appendage (sloshing effect);
- Base of the robotic arm;
- 7 segments of the robotic arm - each segment has a revolute joint (rotation only around one axis).

For the following chapters the Chaser platform is considered to be composed by the hub, fuel and oxidizer tanks, and the base of the manipulator.

Each substructure is represented by a TITOP block with two channels. All of these elements, except for the hub, have an inverse and a direct dynamic channels (defined in section 3.1.2).

In figure 3.6, a schematic of the TITOP blocks listed above and the respective connections is presented. However, the rotation matrices \(R\) between the blocks are not represented for sake of simplification.
Figure 3.6: Implementation of the TITOP model, which represents the dynamics and kinematics of the Chaser.

The hub is a special element because it is represented by a TITOP block with two inverse dynamic channels (forces and torques applied by the child and the parent are inputs). Another special feature is that it is the only body with more than 1 Child block. The children of the hub are the fuel tank, the oxidizer, and the base of the robotic arm.

The parent of the hub is considered to be its surroundings. Therefore, the forces and torques applied on this body are environmental disturbances described in chapter 2, as well as the resulting forces and torques provided by the thrusters. For this, the connection point \( P \), i.e. the point where its parent applies the commands, is set to the hub CoM (point \( \lambda \)), which is constant in the body frame.

Both the fuel tank and the oxidizer are considered flexible appendages given to the sloshing effects inside the tanks, which can be roughly modelled as mass-spring system (as previously presented). Both tanks were divided into three regions with different sloshing masses and frequencies.

An important feature of the TITOP model is the freedom in the initialization of the blocks. The choice of the reference frame relative to each substructure is not important in terms of results, considering that the initialization and the transformations are correctly performed. However, it can ease the resolution of the problem.

The hub fixed body frame corresponds to frame \( F_C \) and it was previously defined, in chapter 2. The base body frame \( F_0 \) is centred at the connection point between the hub and the base, and it is rotated \(-90^\circ\) around the axis \( x_{CGFF} \).

For a robotic arm, a systematic choice of reference frames is the Denavit Hartenberg Representation, also called DH convention. This convention specifies a set of rules to define the axis of the reference
frame. Besides the consistency in the reference frames definition, another advantage is its inherent computational efficiency. This is explained by the fact that the only variable parameter of a DCM (Direction Cosine Matrix) of some segment \( i \) is \( \theta_i \).

There are 2 conventions associated to the Denavit Hartenberg Representation: the classical and the modified. The reader is now referred to Appendix A where both approaches and their differences are explored.

The convention selected for this project was the modified DH convention. Here, axis \( z_i \) is defined along the axis of revolution of joint \( i \), where \( \theta_i \) is the revolute angle and \( i \in \{1 \ldots 7\} \).

The DH parameters in Table 3.1, provided in [18], resulted in the reference frames presented in Figure 3.7.

When a command torque is applied at joint \( i \), the total angular acceleration of body \( i \) changes, hence so does the joint angle \( \theta_i \).

Since \( \theta_i \) changes, the respective transformation matrix \( R \), in equation (3.9) and given by A.6, must be updated.

It should be noted that the freedom in the initialization of the TITOP blocks allows us to define mass and inertia matrices for each substructure as uncertain parameters, using the Robust Control Toolbox. Thus, it is possible to define the state space of each TITOP block as an uncertain state space.

The major uncertainty in the system is due to flexible elements. The fuel tank and oxidizer looses mass, therefore the mass of the entire system is not constant and neither is its center of mass. The sloshing masses and frequencies are also uncertain.
3.3 Model Validation

As described in section 3.1.2, the TITOP model assumes low angular rate, thus Centrifugal and Coriolis accelerations are neglected. To a better understanding of the effects of neglecting these non-linear terms, a series of tests were performed.

The response of a TITOP model is compared to a response of an equivalent valid model, built in Simscape environment and where no assumptions regarding low angular rate are made.

In order to clearly evaluate the results and to have a more intuitive understanding of the dynamics of the model, a simpler model was built. This model corresponds to a system with two masses at the extremities, and two links in between - see figure 3.8.

Then, a test using a structure similar to the Chaser was performed. This structure is composed by a hub, a robotic arm with 7 segments and a small base for the manipulator. These substructures were set equal to the respective Chaser substructures. After testing this structure, it was possible to understand the impact of the accumulation of error, due to the linearisation effects.

Tests with a simple model

The simple model, represented in figure 3.8a, is composed by two bodies at the extremities (numbers 1 and 2), which have similar mass and inertia. Besides, the structure has two segments (numbers 3 and 4), each with a revolute joint represented by a red ball. These segments have nearly no inertia and very small mass. Thus, these two segments work has two links, i.e. they just transfer the dynamics from body 1 to body 2 and vice versa, without changing the dynamic motion of the whole structure significantly.

The moments of inertia and mass set for the 4 elements are presented in table 3.2, whereas the products of inertia (off diagonal elements of the inertia tensor) are zero.

<table>
<thead>
<tr>
<th>Body</th>
<th>Mass [kg]</th>
<th>Inertia [kg/m²]</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$I_{xx}$ $I_{yy}$ $I_{zz}$</td>
</tr>
<tr>
<td>1</td>
<td>30</td>
<td>30 30 30 30</td>
</tr>
<tr>
<td>2</td>
<td>30</td>
<td>30 30 30 30</td>
</tr>
<tr>
<td>3</td>
<td>0.01</td>
<td>0.01 0.01 0.01 0.01</td>
</tr>
<tr>
<td>4</td>
<td>0.01</td>
<td>0.01 0.01 0.01 0.01</td>
</tr>
</tbody>
</table>

Table 3.2: Mass and Inertia of the substructures of the body in figure 3.8a.

It should be noted that the inputs to the Simscape model are mass and the inertia. The dimensions set for the blocks are only a matter of representation. In fact, the inertia of the manipulator segments do not correspond to a parallelepiped, but instead corresponds, for example, to a sphere or a cube. A different graphical representation was preferred in order to depict the motion of the system more clearly.

In figure 3.8, we can see the response of the system to a torque applied on the CoM of Body 1, around the $z$ axis of its body frame ($T_z = 5N\text{m}$, for $\sim 3s$).

Next, follows the test results from different commands applied at the structure. Here, the initial state of the system is defined with null velocity and the inertial frame coincides with the body frame at $t = 0s$. 
Test with a torque of $\vec{T}_{P/A,P} = 5\hat{z}Nm$ applied on the Body 1

A torque of $\vec{T}_{P/A,P} = 5\hat{z}Nm$ is applied at Body 1, on its center of mass (in the TITOP block, point A is matched to point P).

Given the geometry of Body 1 (off diagonal elements of the inertia matrix are null) and that the torque only actuates around $z$ axis, the angular acceleration at Body 1 must be zero except for the latter axis - see figure 3.9. In addition, the non linear term of the Torque equation (3.6) - $\omega \times (I\omega)$ - is also zero. Hence, the divergent results between the valid and the TITOP models would only be due to Centrifugal and Coriolis fictitious forces. In Body 1 rotating frame, its connection point C or any other point of that body is not moving, thus $\vec{i}^C$ in (3.3) is zero. Thus, Coriolis force is also zero!

Figure 3.9: Inertial angular acceleration at Body 1. Response to a constant torque of $\vec{T}_{P/A,P} = 5\hat{z}Nm$ applied at Body 1. The results of the model developed in Simscape are overlapped with the results from the TITOP model; the results from Simscape model are represented in full line, and the ones from the TITOP model in dashed line.

Kinematics are transferred from parent block to child block, until it reaches the final block. Because the segments have very small mass, when compared to Bodies 1 and 2, the force that Body 1 applies on the first segment (its child) is approximately the force that Body 1 applies on Body 2 ($\vec{F}_{A/P} = \vec{F}_{C/A} - m(\vec{a}_P + \vec{\omega} \times \vec{P}A) \approx \vec{F}_{C/A}$).
The time series of the inertial linear acceleration at Body 1 CoM is represented in figure 3.10. The $z$ component of the linear acceleration is null because there are no forces being applied in that direction. The other components are due to the response of Body 2 to the torque applied at Body 1 and which is then propagated back to the latter.

In Figures 3.10a and 3.10b, the impact of the linearisation effects are clear.

It should be noted that more substructures the TITOP model has, the greater the error accumulation.

Figure 3.10: Inertial linear acceleration of Body 1. Response to a constant torque of $\vec{T}_{A/P, P} = 5\hat{z}Nm$ applied at Body 1. The results of the model developed in Simscape are overlapped with the results from the TITOP model.

Comparing Figure 3.10a to Figure 3.11, it is noticeable that, the higher the angular velocity, the higher is the difference between the valid model and the TITOP model, as expected.

Due to integration effects the angular velocity error is delayed.

Figure 3.11: Angular velocity of Body 1, about $z$ axis ($\omega_z$). Response to a constant torque of $\vec{T}_{A/P, P} = 5\hat{z}Nm$ applied at Body 1. The results of the model developed in Simscape are overlapped with the results from the TITOP model.

Figure 3.11: Angular velocity of Body 1, about $z$ axis ($\omega_z$). Response to a constant torque of $\vec{T}_{A/P, P} = 5\hat{z}Nm$ applied at Body 1. The results of the model developed in Simscape are overlapped with the results from the TITOP model.

Regarding joint angles, the results were quite similar to the Simscape model after 3 s; with an absolute error of 8° for the first joint and 9° for the second joint. The diverging results of the angular acceleration are delayed when integration takes place. Hence, the angle error at the joints...
is smaller than the angular acceleration error, which corresponds, in both joints, to 13 °/s² - see Figures B.1 and B.2, from Appendix B.

- **Test with a torque of** $\vec{C}_m = 0.1\hat{y} Nm$ **applied at the first joint**

In this test, the revolute joints only rotate around the $z$ axis. The results obtained for joint angular accelerations and joint angles, when a torque is actuating on the first joint, are represented in figures 3.12 and 3.13, for the 1st and 2nd joints respectively. A small torque was applied due to the small inertia on the segments.

We can see from the Figures below that the segments motion in the TITOP model follows the motion in the Simscape model. As noted before, the divergence in the angular acceleration is delayed in the angles because of the double integration.

![Angular Acceleration at Joint 1](image1)  ![Angle at Joint 1](image2)

(a) Angular acceleration at the joint ($\dot{\theta}_1$).  (b) Angle at the joint ($\theta_1$).

Figure 3.12: Angular acceleration and angle at the first joint. Response to a constant torque of $\vec{T}_{A/P} = 0.1\hat{z}$ applied at joint 1. The results of the model developed in Simscape are overlapped with the results from the TITOP model.

![Angular Acceleration at Joint 2](image3)  ![Angle at Joint 2](image4)

(a) Angular acceleration at the joint ($\dot{\theta}_2$).  (b) Angle at the joint ($\theta_2$).

Figure 3.13: Angular acceleration and angle at the second joint. Response to a constant torque of $\vec{T}_{A/P} = 0.1\hat{z}$ applied at joint 1. The results of the model developed in Simscape are overlapped with the results from the TITOP model.
**Tests with a 7 segments-robotic arm**

Tests with the hub and the full length robotic arm were performed in order to evaluate the **rigid dynamics** of the model when comparing to an analogous non linear model developed in *Simscape* (flexible components were not considered).

With a robotic arm set accordingly to the parameters in table 3.1, the system obtained is as depicted in Figure 3.14.

![Figure 3.14: Representation of a body composed with the hub and robotic arm of the Chaser.](image)

The following plots correspond to the response of the system to an external torque $\vec{T}_{A/P} = 80 Nm$ applied at the hub CoM and depicted in Figure 3.15. This torque is applied for only 0.2 s, resulting in a constant angular velocity for the next 2.8s. The impulse magnitude was not a random choice, since it produces a slightly higher angular velocity than the Target tumbling rate - see Figure 3.16.

![Figure 3.15: External torque applied at the hub, in Nm.](image)

Figures 3.17 and 3.18 show the angular and linear accelerations of the Chaser. Due to fictitious centrifugal acceleration, linear acceleration $a_x$ is not null, in the Simscape model. However, for a tumbling rate of $\omega \approx 0.05$ rad/s the absolute error of both linear and angular accelerations are very small - see Table 3.3.

<table>
<thead>
<tr>
<th>Linear Acceleration Error [ m/s²]</th>
<th>Angular Acceleration Error [ o/s²]</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_x$ $a_y$ $a_z$ $\dot{\omega}_x$ $\dot{\omega}_y$ $\dot{\omega}_z$</td>
<td></td>
</tr>
<tr>
<td>$3.9 \times 10^{-5}$ $5.0 \times 10^{-6}$ $1.4 \times 10^{-5}$ $8.1 \times 10^{-4}$ $7.3 \times 10^{-4}$ $2.1 \times 10^{-3}$</td>
<td></td>
</tr>
</tbody>
</table>

Table 3.3: Absolute error of linear and angular accelerations at the hub CoM.

Finally, to better understand the effects of neglecting squared terms of the angular rate in the system dynamics and kinematics, a command torque of $C_m = 10 Nm$ (half the nominal back-drive torque at a joint) is applied at the first Joint - see the impulse command in Figure 3.19. The mass and inertia are much higher in the hub than in the manipulator, hence the perturbation on the hub is very small, as depicted in Figures B.3 and B.4 that show the linear and angular accelerations, respectively.
Figure 3.16: The hub angular rate. Response to an external torque actuated on the hub CoM. The plots in full line correspond to the Simscape Model and the plots in dashed line to the TITOP model.

Figure 3.17: Angular acceleration at the hub CoM. Response to an external torque actuated on the hub CoM. The plots in full line correspond to the Simscape Model and the plots in dashed line to the TITOP model.

The responses of the first and last joints of the robotic arm are represented in Figure 3.20. The absolute error of the angular acceleration at the 1st joint is 0.28 °/s² and at the last joint 0.99 °/s². The error at the last joint is higher; as a consequence, the corresponding resulting joint angle using the TITOP model quickly diverges from the result of the Simscape model, when compared to the 1st joint. When using the TITOP model to represent the Chaser, command torques at the joints $C_m$ should be small, thus, causing a slow motion at the End Effector (EE) of the manipulator.

The absolute error of the angular acceleration at each joint are presented in Table 3.4.

From the results of both tests, we can conclude that this model is representative of the Chaser dynamics. Therefore, it will be introduced in the control systems, while aware of the limitations of neglecting squared
Figure 3.18: Linear acceleration at the hub CoM. Response to an external torque actuated on the hub CoM. The plots in full line correspond to the Simscape Model and the plots in dashed line to the TITOP model.

Figure 3.19: Command applied at the 1st joint.

terms of the angular rate. Later in chapter 5, it is explained how to resolve the non linearities due to the time-varying transformation matrices.

| Absolute Error of the Joints Angular Acceleration [°/s²] |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| $\dot{\theta}_1$ | $\dot{\theta}_2$ | $\dot{\theta}_3$ | $\dot{\theta}_4$ | $\dot{\theta}_5$ | $\dot{\theta}_6$ | $\dot{\theta}_7$ |
| 0.28            | 0.35            | 0.24            | $4.9 \times 10^{-3}$ | 0.92            | 0.20            | 0.99            |

Table 3.4: Absolute error of the angular acceleration, at each joint of the manipulator.
Figure 3.20: Plots of the angular acceleration, angular rate and joint angles at the first and last segment. Response to a command torque $C_m = 10$Nm acting on the 1st joint. The plots in full line correspond to the Simscape Model and the plots in dashed line to the TITOP model.
Chapter 4

Inverse Dynamics and Kinematics

The aim of this project is to capture another satellite while maintaining the relative pose of the Chaser w.r.t. the Target. Therefore, one needs to command forces and torques that will actuate on the hub and command torques on the joints of the manipulator. These commands are calculated using the state of the system: position, linear and angular velocities, and attitude of the hub, as well as joint angles and joint rates.

In this chapter, some important background to compute these commands is presented. First, the Inverse Kinematic Problem (IKP) is introduced, then Dynamics Coupling between the platform and the robotic arm is presented. The chapter ends with the section Trajectory Tracking and Control of the spacecraft, where strategies to control the spacecraft are presented.

4.1 Inverse Kinematic Problem

The Inverse Kinematic Problem (IKP) is concerned with what joint angles are required to reach the desired pose, and it is defined by equation

$$\theta = f(x),$$

where $x \in \mathbb{R}^m$ is the task space and $m$ is the dimension of the task space; $\theta \in \mathbb{R}^n$ corresponds to joint angles of some manipulator, with $n$ joints.

Linear and angular velocities can be related to joints angular rate by [12]

$$\dot{x} = J\dot{\theta},$$

(4.1)

where $\dot{x} \in \mathbb{R}^m$ is, in this case, the linear and angular velocities of the EE, $\dot{\theta} \in \mathbb{R}^n$ represents the joint rates. The Jacobian matrix $J \in \mathbb{R}^{m \times n}$ maps the spacial velocity of the EE into joints rate [10].

However, we are interested in obtaining joint rates using the known angular and linear velocities of the end effector. Hence, we have

$$\dot{\theta} = J^{-1}\dot{x}.$$  

(4.2)
It is remarked that the Jacobian can be inverted when the manipulator Degrees of Freedom (DoFs) are equal to the dimension of the task space \( n = m \) and it not at a singular point \( (J \) has full row rank). Hence, \( J \) cannot be inverted when [29, 30]:

1. two revolute axes are parallel and coincident. If two axes are parallel but not coincident the Jacobian is not singular
2. two prismatic axes are parallel
3. the manipulator has less than 6 DoFs
4. the manipulator has more than 6 DoFs

The 1st situation may occur when there are at least 2 revolute joints; and the 2nd when there are at least 2 prismatic joints. In cases 3 and 4, the Jacobian matrix is not squared, hence it is not invertible.

In the context of this thesis, a robotic arm with 7 revolute joints is considered. Therefore, we are only concerned with the 1st and 4th cases.

Redundant manipulators have more internal DoFs than the required to perform a specified task, \( i.e. n > m \). The Chaser has a 7 jointed robotic arm for a 6 DoFs problem, thus this satellite has a redundant manipulator.

These kind of manipulators have enough DoFs to perform the desired task while constraining the workspace. Examples of constraints are [30]:

- obstacle avoidance
- minimization of torque perturbations
- singularity avoidance
- bounded joint velocities

Constraints can be used to execute smooth trajectories as well as to avoid solutions with no physical meaning. The IKP for a redundant manipulator has multiple solutions. If proper constraints are enforced, a more efficient EE trajectory could be obtained.

Before proceeding to different methods of solving the IKP, the derivation of the Jacobian is taking place.

### 4.1.1 Jacobian Derivation

Firstly, a small introduction to Homogeneous Transformation Matrices is made.

The position and orientation of a manipulator can be obtained from the hub pose using the Homogeneous Transformation Matrix, \( T \in \mathbb{R}^{4 \times 4} \), or just Transformation Matrix for language simplicity.

The Transformation Matrix from frame \( \mathcal{F}_i \) to frame \( \mathcal{F}_j \) is given by

\[
T^j_i = \begin{bmatrix} R^j_i & O^j_i \\ 0 & 1 \end{bmatrix},
\]  

(4.3)
where \( \mathbf{R}_i^j \in \mathbb{R}^{3 \times 3} \) is a Rotation Matrix that expresses the orientation from frame \( \mathcal{F}_i \) to frame \( \mathcal{F}_j \), \( \mathbf{O}_i^j \in \mathbb{R}^3 \) is the coordinate vector of the origin of frame \( \mathcal{F}_i \) written in frame \( \mathcal{F}_j \), and \( \mathbf{0} \in \mathbb{R}^3 \) is a null vector. Throughout this chapter variables in bold correspond to vectors or matrices, while unbolded terms are scalars.

The rotation and translation of some vector \( \mathbf{v} \in \mathbb{R}^3 \) from \( \mathcal{F}_i \) to \( \mathcal{F}_j \) is given by

\[
\begin{bmatrix}
\mathbf{v}' \\
1
\end{bmatrix} = \mathbf{T}^j_i
\begin{bmatrix}
\mathbf{v} \\
1
\end{bmatrix},
\]

where \( \mathbf{v}' \in \mathbb{R}^3 \) is the transformed vector.

This transformation can also be obtained by multiplying subsequent Transformation Matrices, such as

\[
\mathbf{T}^j_i = \mathbf{T}^j_{j+1} \cdots \mathbf{T}^{i-1}_i,
\]

where the Transformation Matrices \( \mathbf{T}^{i-1}_i \) is given by

\[
\mathbf{T}^{i-1}_i =
\begin{bmatrix}
\cos(\theta_i) & -\sin(\theta_i) & 0 & a_i \\
\sin(\theta_i) & \cos(\theta_i)\cos(\alpha_i) & -\sin(\alpha_i) & -d_i\sin(\alpha_i) \\
\sin(\theta_i) & \cos(\theta_i)\sin(\alpha_i) & \cos(\alpha_i) & d_i\cos(\alpha_i) \\
0 & 0 & 0 & 1
\end{bmatrix},
\]

where \( \theta_i, \alpha_i, a_i, \) and \( d_i \) are the Modified DH parameters presented in Appendix A.1. For more information about the DH representation and the above equation the reader is referred to Appendix A. The dashed lines are delimiting the Rotation and Translation Matrices, \( \mathbf{R}^{i-1}_i \) and \( \mathbf{O}^{i-1}_i \) respectively.

As an example, to perform the transformation from the last segment of the robotic arm (frame \( \mathcal{F}_7 \)) to its base (frame \( \mathcal{F}_0 \)), one must perform the following step

\[
\mathbf{T}^0_7 = \mathbf{T}^0_1 \mathbf{T}^1_2 \cdots \mathbf{T}^5_6 \mathbf{T}^6_7.
\]

It should be noted that, if using the DH convention, the only variable parameter in equation 4.5 is the joint angle \( \theta_i \), corresponding to joint \( i \).

Knowing the Transformation Matrix from the EE to the hub, its pose can be obtained w.r.t. to the Chaser body frame \( \mathcal{F}_C \) (defined in chapter 2).

The frame centred on EE and with its angular rate is defined as \( \mathcal{F}_{EE} \). It as the same angular rate as \( \mathcal{F}_7 \) and \( \mathbf{O}_{EE}^7 \) is constant. Therefore, \( \mathbf{T}_{EE}^7 \) is constant. The reference frames of each link of the robotic arm and obtained with the Modified DH convention are represented in Figure 3.7.

Note that we can easily obtain the EE pose w.r.t. the Chaser body frame since the Transformation Matrix from \( \mathcal{F}_0 \) to \( \mathcal{F}_C \) is constant and known, which yields

\[
\mathbf{T}_{EE}^C = \mathbf{T}_0^C \mathbf{T}^0_7 \mathbf{T}_{EE}^7.
\]
The Jacobian maps the velocity of the EE into joint rates. Here, we are only concerned with the kinematics of the robotic arm due to joints motion, thus the influence of the base on the robotic arm is not considered. The Jacobian $J$ can be divided into the Jacobian for linear velocity $J_v$ and the Jacobian for angular velocity $J_\omega$:

$$J = \begin{bmatrix} J_v \\ J_\omega \end{bmatrix}.$$ 

This derivation considers all joints revolute (i.e. the variable DH parameters are the joint angles $\theta$). For prismatic joints, the derivation of the Jacobian is presented in [31].

According to the modified DH convention, the angular rate of joint $i$ w.r.t. the previous segment frame $\mathcal{F}_{i-1}$ is given by

$$\dot{\theta}_i^{i-1} = \dot{\theta}_i \mathbf{R}_i^{i-1} \mathbf{k},$$

where $\mathbf{R}_i^{i-1}$ is the Rotation Matrix from $\mathcal{F}_i$ to $\mathcal{F}_{i-1}$ given in equation (4.5) and $\mathbf{k}$ is a vector basis of the Euclidean space $\mathbb{R}^3$ and it is given by $\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$ and it represents the axis of rotation of joint $i$ ($i \in [1, n]$).

Hence, the EE angular velocity w.r.t the satellite body frame $\mathcal{F}_C$ can be obtained by

$$\omega_{EE}^C = \sum_{i=1}^{n} \dot{\theta}_i \mathbf{R}_i^C \mathbf{k} = \sum_{i=1}^{n} \dot{\theta}_i z_i^C,$$

where $\mathbf{R}_i^C$ is the Rotation Matrix from frame $\mathcal{F}_i$ to frame $\mathcal{F}_C$ and given by

$$\mathbf{R}_i^C = \mathbf{R}_0^C \mathbf{R}_1^C \cdots \mathbf{R}_{i-1}^C.$$

The ith column of the Jacobian for the angular velocity is, thus, expressed as:

$$J_{\omega_i} = z_i^C.$$ (4.8)

Using equation (3.2) for the linear velocity, we obtain the following expression for the EE, w.r.t. frame $\mathcal{F}_C$:

$$v_{EE}^C = \sum_{i=1}^{n} \dot{\theta}_i \times r_{i \rightarrow EE}^C,$$ (4.9)

where $r_{i \rightarrow EE}^C \in \mathbb{R}^3$ is the vector from joint $i$ to the EE; $\dot{\theta}_i^C$ is the angular rate of joint $i$ written in frame $\mathcal{F}_C$ and yields

$$\dot{\theta}_i^C = \dot{\theta}_i z_i^C.$$ 

Vector $r_{i \rightarrow EE}^C$ is obtained by the vectorial difference given by

$$r_{i \rightarrow EE}^C = O_{EE}^C - r_i^C.$$

the position of the EE in frame $\mathcal{F}_C$ is represented by $O_{EE}^C \in \mathbb{R}^3$ and it can be obtained by subsequent transformation matrices multiplication in equation (4.7). The position of joint $i$ in the same frame is
obtained by
\[ r_i^C = O_i^C + R_i^C r_i, \]
where \( r_i \in \mathbb{R}^3 \) is the position of the joint in its body frame \( \mathcal{F}_i \).

To include the cross product of equation (4.9) in a matrix, one benefits of the skew symmetric matrix defined by equation (3.17). Hence, we can write the \( i \)th column of the Jacobian for the linear velocity as
\[ J_{v_i} = (z_i^C \times (O_{EE}^C - r_i^C)). \]
(4.10)

where \((z_i^C \times) \in \mathbb{R}^{3 \times 3}\) is the Skew Symmetric matrix of \( z_i^C \).

### 4.1.2 Pseudo-inverse Method

The Jacobian matrix for redundant manipulators is not square, thus it cannot be inverted. In this case, one has to make use of the pseudo-inverse (or Moore-Penrose inverse) of \( J \), i.e. \( J^\dagger \in \mathbb{R}^{n \times m} \) and yields
\[ \dot{\theta} = J^\dagger x. \]
(4.11)

The pseudo-inverse gives the best possible solution in the sense of least squares, by minimizing the cost function given by [32]
\[ \Phi(\dot{\theta}) = \|J\dot{\theta} - x\|^2. \]
(4.12)

To find the set of joint rates that minimize \( \Phi(\dot{\theta}) \), one shall find the partial derivative of equation (4.12) for the unknown \( \dot{\theta} \) and set it equal to zero, which yields
\[ \dot{\theta}_{\text{min}} = (JJ^T)^{-1}J^T. \]

When \( J \) has full row rank, the minimum magnitude solution \((JJ^T)^{-1}\) exists and the pseudo-inverse is given by
\[ J^\dagger = (JJ^T)^{-1}J^T. \]
(4.13)

The pseudo-inverse calculation method can also be obtained using the Singular Value Decomposition (SVD) of \( J \), which is expressed by
\[ J^\dagger = U\Sigma^\dagger V^T = \Sigma_{i=1}^{n} \frac{1}{\sigma_i} u_i v_i^T, \]
(4.14)

where \( \Sigma, V \) and \( U \) are obtained from the SVD of \( J \); \( \sigma_i \) corresponds to the singular value of \( J \), and \( u_i \) and \( v_i \) to columns of \( U \) and \( V \) respectively [12, 30, 33].

\( J^\dagger \) becomes singular when a singular value approaches zero, and it no longer has full row rank. Hence, there is a discontinuity in the pseudo-inverse, resulting in large values of joint angles \( \theta \). Large values of \( \theta \) amplifies numerical errors on robot positioning [33].

Given that \(|\det(J)| = \sigma_1 \ldots \sigma_n\), one can determine whether we are close to singularity if \( \det(J) \) is close
to zero.

It should be noted that when using the pseudo-inverse in a redundant manipulator, the extra DoFs are not being exploited; the pseudo-inverse only provides the primary solution [12]. However, for redundant manipulators one can make use of the null space of the Jacobian, which corresponds to the group of angular rates $\theta_{\mathcal{N}}$ which do not generate any velocity $\dot{x}$. The null space of the Jacobian is defined by:

$$\mathcal{N}(J) = \{ \forall \theta_{\mathcal{N}} \in \mathbb{R}^n : J^T \theta_{\mathcal{N}} = 0 \}.$$ 

A property of the pseudo-inverse method is that matrix $(I - J^T J)$ performs a projection onto the null space of $J$. Hence, the IKP can be solved by

$$\dot{\theta}_{\text{min}} = J^T \dot{x} + (I - J^T J) \gamma,$$

for some random vector $\gamma \in \mathbb{R}^n$.

As an example, in [34], Yoshikawa uses the concept of manipulability measure to avoid singularities. Singularities are not desirable because the manipulator cannot move in a certain direction, meaning that the manipulability is deteriorated. Yoshikawa proposed the quantity manipulability measure given by $w = \sqrt{\det(JJ^T)}$ and uses it for determining the best postures of various types of manipulators.

4.1.3 Damped Least Squares Method

According to [12], generating a velocity component in certain directions at the EE and close to a singular posture, requires very high joint rates, which are not affordable by the manipulator.

In [33], different approaches to avoid singularities are presented. Reducing the global gain of the pseudo-inverse is not a good approach since it damps the gain in directions the robot should be moving. Hence, the Jacobian Damping method is proposed. Here, a small diagonal term $\lambda$ is added to the pseudo-inverse. $\lambda$ varies according to the proximity to singularity which can be measured by $\det(J)$. However, it should be noted that the effect might not be enough to avoid singularities.

In [32, 33], a damped least squares method is proposed, in order to avoid the pseudo-inverse singularity problems. This method only provides an approximate solution to the IKP, but offers satisfactory results regarding singularity avoidance.

A weighted norm of the joint rates is added so that high joint rates are penalized. Thus, the cost function is given by

$$\Phi(\dot{\theta}) = \| J \dot{\theta} - \dot{x} \|^2 + \| \lambda \dot{\theta} \|^2,$$

(4.15)

$\lambda$ is the damping factor and should be set high when close to singularities.

The set of joint rates that minimize $\Phi(\dot{\theta})$ can be obtained by equation (4.16).

$$\dot{\theta}_{\text{min}} = (J^T J + \lambda^2 I)^{-1} J^T \dot{x}.$$

(4.16)

In addition, the singular values of $(J^T J + \lambda^2 I)^{-1} J^T$ are given by

$$\frac{\sigma_i}{\sigma_i^2 + \lambda^2}.$$
Note that if $\lambda = 0$, equation (4.16) becomes $J^\dagger$, given in (4.13), with singular values $\frac{1}{\sigma_i}$.

The damping factor $\lambda$ can be constant or variable. It should be large enough so that the solutions are well behaved near singularities, but if too large the convergence rate is too slow. If $\lambda$ is constant, the implementation is simpler and has less computational burden. However, its value might not be enough to avoid numerically instabilities. In [35], a singular region and a maximum damping factor are defined, and then the damping factor is obtained according some law, which is function of the smallest singular value of $J$.

In Figures 4.1 - 4.4 responses of the Chaser to command torques at the joints are presented. Different methods to solve the IKP were used: first with $\lambda = 0$, i.e. using the pseudo-inverse, then with constant $\lambda = 0.2$ using the damped least squares method. The platform is considered to have fixed position and attitude at all time.

Firstly, a feasible goal position for the EE is considered - see Figures 4.1 and 4.2. In both cases, the manipulator can reach the desired point. However, the method that considers $\lambda \neq 0$ in equation (4.16) causes a slightly smaller disturbance on the hub - compare Figures 4.1a and 4.1b to Figures 4.2a and 4.2b. While the manipulator moves, the hub tries to keep the same pose. The higher the joint rates, the higher the disturbances at the hub, thus the more difficult it is to keep its pose. These disturbances are expressed through linear and angular accelerations at the hub, which are later converted to command forces and torques.

![Time Series of the Linear Acceleration at the Hub w/ $\lambda = 0$](chart1.png)

**Figure 4.1:** Response of the Chaser to external commands, using the pseudo-inverse method to solve the IKP. A feasible goal position is considered.

When commanding the robotic arm to a non-feasible point, e.g. where the length of the robotic arm is not enough to reach it, the responses using the same methods are quite divergent - see Figures 4.3 and 4.4.

It is clearly shown that the motion when $\lambda \neq 0$ is much smoother and does not lead to singularities. As described in [12], if close to a singular pose, high joint rates are required - which are not affordable.
Figure 4.2: Response of the Chaser to external commands, using damped least squares method to solve the IKP. A feasible goal position is considered.

by the manipulator. In Figures 4.3c, joints angular acceleration explode, leading to high joint rates. In contrast, in Figure 5.16c, which correspond to joints angular acceleration when $\lambda = 0.2$, a much slower transition is noticeable. Leading to a smoother motion of the EE.

Even though the robotic arm cannot reach the position, it stabilizes in a position with a constant error with respect to the goal position.

Figure 4.3: Response of the Chaser to external commands, only using the Pseudo Inverse to solve the IKP. A non-feasible goal position is considered.

The damped least squares method is able to reach the goal position, within the physical limitations of the arm, and without adding appreciable computational effort. When a non-feasible goal is considered, the motion of the arm is still smooth, thus keeping the integrity of the system intact.

These results were achieved with only a constant $\lambda$. Due to its simplicity and efficiency, this is the approach to be implemented in the control system.
Figure 4.4: Response of the Chaser to external commands, using the singularity avoidance method to solve the IKP. A non-feasible goal position is considered.

Besides the damped least squares method, there is a whole world when using the null space of the Jacobian to introduce certain constrains. In [11], obstacle avoidance and joint limit avoidance algorithms are presented. In this dissertation, we are only concerned with potential numerical instabilities due to singularities. Therefore, no extra constrains are added do the system, using the null space of $J$.

### 4.2 Inverse Dynamics

The dynamics of a satellite composed by a platform and a robotic arm can be represented by equation [4, 8]

\[
\begin{bmatrix}
H_p & H_{pm} \\
H_{pm}^T & H_m
\end{bmatrix}
\begin{bmatrix}
\ddot{x}_H \\
\ddot{\theta}
\end{bmatrix}
+ C(\theta, \dot{\theta}) + g =
\begin{bmatrix}
F_p \\
\tau
\end{bmatrix}.
\]  

(4.17)

Remember that the platform is composed by the hub, the fuel and oxidizer tanks, and the base of the manipulator.

The notation used in (4.17) is listed below, considering an $n$-jointed manipulator:

- $x_H \in \mathbb{R}^6$: pose at the hub CoM
- $H_p \in \mathbb{R}^{6 \times 6}$: platform inertia matrix
- $H_{pm} \in \mathbb{R}^{6 \times n}$: coupling inertia matrix, between the platform and the manipulator
- $H_m \in \mathbb{R}^{n \times n}$: manipulator inertia matrix
- $F_p \in \mathbb{R}^6$: external forces and torques actuating on the platform
- $\tau \in \mathbb{R}^n$: external torques actuating on the joints of the robotic arm
- $C \in \mathbb{R}^{6+n}$: accounts for the non linear terms, e.g. Coriolis and centrifugal forces
\( g \in \mathbb{R}^{6+n} \): gravitational force and torque

It should be noted that equation (4.17) is non linear. Since the goal is to design a linear controller, the linearised version of (4.17) must be considered. In addition, gravitational torques are neglected and gravitational forces are introduced in the control system as disturbances. Thus the term \( g \) also disappears, which yields:

\[
\begin{bmatrix}
H_p & H_{pm} \\
H_{pm}^T & H_m
\end{bmatrix}
\begin{bmatrix}
\ddot{x}_p \\
\dddot{\theta}
\end{bmatrix}
\approx
\begin{bmatrix}
F_p \\
\tau
\end{bmatrix}.
\]

(4.18)

The kinematic relation among the pose of the EE \( x_{EE} \), the pose of the hub \( x_p \), and joint angles \( \theta \) is given by [8]:

\[
\dot{x}_{EE} = J_m \dot{\theta} + J_p \dot{x}_p,
\]

(4.19)

where \( J_m \in \mathbb{R}^{6 \times n} \) is the manipulator Jacobian derived in section 4.1.1, and \( J_p \in \mathbb{R}^{6 \times 6} \) is called the platform Jacobian and it is given by

\[
J_p = \begin{bmatrix}
I & (r_{H \rightarrow EE}) \\
0 & I
\end{bmatrix},
\]

(4.20)

where \( (r_{H \rightarrow EE}) \in \mathbb{R}^{3 \times 3} \) is the skew symmetric matrix of the vector from the hub CoM to the EE.

The elements of the inertia matrix \( H \) are going to be presented below.

**Manipulator Inertia Matrix**

To obtain the robotic arm inertia matrix \( H_m \) we shall express the problem using the Lagrange-Euler Formulation. The solution can also be obtained using Newton mechanics. However Lagrangian mechanics eases the formulation of a problem that considers a body with multiple variables, e.g. a robotic arm.

The Lagrange-Euler Formulation is based on two equations and it is expressed by

\[
\begin{cases}
\mathcal{L} = \mathcal{K} - \mathcal{P} \\
\tau = \frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{\theta}} \right) - \frac{\partial \mathcal{L}}{\partial \theta}
\end{cases}
\]

(4.21)

In (4.21), \( \mathcal{L} \in \mathbb{R} \) is the non relativistic Lagrangian, \( \mathcal{K} \in \mathbb{R} \) and \( \mathcal{P} \in \mathbb{R} \) are the Kinetic and Potential energies respectively, \( \dot{\theta} \in \mathbb{R}^n \) is the generalized robot coordinates (joint angles), and \( \tau \in \mathbb{R}^n \) is the generalized torque at the arm joints, and written in an inertial frame \( F_I \).

The Kinetic energy of some body is given by [30]

\[
\mathcal{K} = \frac{1}{2} m (v^3)^T v^3 + \frac{1}{2} (\omega^3)^T I^3 \omega^3,
\]

where \( m \in \mathbb{R} \) is the mass of the body, \( v^3, \omega^3 \in \mathbb{R}^3 \) are the inertial linear and angular velocities at the CoM, and \( I^3 \in \mathbb{R}^{3 \times 3} \) is the inertia tensor, which can be written as a function of the inertia tensor \( I^i \) defined around the CoM of the body an w.r.t. the body frame. In order to do so, one needs to rotate the tensor.
Some tensor $I^i$ can be rotated using the expression below

$$I^3 = R^3_i I^i (R^3_i)^T,$$  \hspace{1cm} (4.22)

where $R^3_i \in \mathbb{R}^{3 \times 3}$ is the rotation matrix obtained from the Homogeneous Transformation Matrix $T^3_i \in \mathbb{R}^{3 \times 3}$.

The robotic arm is composed by multiple segments, thus the total kinetic energy of this element is the sum of the kinetic energy of each segment, which yields

$$K = \sum_{i=1}^{n} \frac{1}{2} m_i (v_i^3)^T v_i^3 + \frac{1}{2} (\omega_i^3)^T I^3_i \omega_i^3.$$  \hspace{1cm} (4.23)

The linear and angular velocities, $v_i^3$ and $\omega_i^3$, in (4.23) can be replaced by a function of the generalized coordinate $\theta$ using the Jacobian. In the last section we obtained the Jacobian that maps joint rates $\dot{\theta}$ to linear and angular velocities of the EE. The same steps can be taken to obtain the Jacobians corresponding to the CoM of each link $i \in [1, n]$.

Thus, the Jacobian for the linear velocity of link $i$ is written as

$$J_{v_i} = \begin{bmatrix} z_1^3 \times (r_{CoM_i}^3 - r_1^3) & \ldots & z_i^3 \times (r_{CoM_i}^3 - r_i^3) & 0_{3 \times (n-i)} \end{bmatrix},$$  \hspace{1cm} (4.24)

where $r_{CoM_i}^3 \in \mathbb{R}^3$ is the position of the CoM of link $i$ w.r.t. an inertial frame $F$, $r_i^3 \in \mathbb{R}^3$ is the position of joint $i$ w.r.t. $F$, and $0_{3 \times (n-i)} \in \mathbb{R}^{3 \times (n-i)}$ is a zero matrix.

The Jacobian for the angular velocity is given by

$$J_{\omega_i} = \begin{bmatrix} z_1^3 & \ldots & z_i^3 & 0_{3 \times (n-i)} \end{bmatrix}.$$  \hspace{1cm} (4.25)

Using equations (4.21)-(4.25), the equation for the kinetic energy of the robotic arm can be expressed as

$$K = \frac{1}{2} \theta^T \sum_{i=1}^{n} \left[ m_i J_{v_i}^T J_{v_i} + J_{\omega_i}^T I_i J_{\omega_i} \right] \theta.$$  \hspace{1cm} (4.26)

From equation (4.26), the equation for the inertia matrix of the robotic arm $H_m$ is obtained.

Two contributions to the potential energy are masses via their gravitational potential energy, and elastic elements via the energy stored when deflected from rest [30]. Here, we are not considering potential energy neither due gravity not due to elastic elements. Thus, we have $P = 0$.

Using equation (4.26 for the Kinetic and applying it to the Lagrangian equations (4.21), we obtain the expression of the generalized joint torque $\tau$, without considering the coupling dynamics between the platform and the manipulator. The derivation of this equation is found in [30] and is given by

$$\tau = H_m \ddot{\theta} + C \dot{\theta},$$
where $C_T$ represents the Coriolis and Centrifugal terms respectively, and $H_m$ is first defined in equation (4.26) as

$$H_m = \left[ m_1 J_{v_1} J_{w_1} + J_{w_1} I_1 J_{v_1} \ldots m_n J_{v_n} J_{w_n} + J_{w_n} I_n J_{v_n} \right].$$  \quad (4.27)

### Platform Inertia Matrix

Similarly to the formulation of the TITOP model in chapter 3, Newton's mechanics are used. The force that acts on the hub equals to the sum of all external forces, thus, according to the second Newton's law we have:

$$F_{I_{tot}}^J = m_{tot} a_{I_{CoM}}^J,$$  \quad (4.28)

where $m_{tot} \in \mathbb{R}$ is the mass of the entire spacecraft and $a_{I_{CoM}}^J \in \mathbb{R}^3$ is the inertial linear acceleration at the CoM of the whole system. The latter can be expressed as a function of the inertial linear acceleration at hub CoM $a_H^J \in \mathbb{R}^3$. The linearised expression for the linear acceleration (also given in Chapter 3) is given by

$$a_{I_{CoM}}^J = a_H^J + \dot{\omega}_{I_{CoM}}^J \times r_{H\rightarrow CoM}^J,$$  \quad (4.29)

where $r_{H\rightarrow CoM}^J \in \mathbb{R}^3$ is the vector from the hub CoM to the entire system CoM; $r_{H\rightarrow CoM}^J \in \mathbb{R}^3$ is written in frame $\mathcal{F}_I$ and it is expressed by

$$r_{H\rightarrow CoM}^J = R_{IC}^J r_{CoM}^{IC} = R_{IC}^J \frac{1}{m_{tot}} \left[ m_{Fuel} r_{Fuel}^{IC} + m_{Ox} r_{Ox}^{IC} + \sum_{i=1}^n m_i r_{CoM_i}^{IC} \right],$$  \quad (4.30)

where $r_{Fuel}^{IC}, r_{Ox}^{IC}, r_{CoM_i}^{IC} \in \mathbb{R}^3$ are the position of the fuel tank CoM, the oxidizer tank CoM, and the CoM of segment $i$ of the manipulator, respectively. These vectors are written in frame $\mathcal{F}_C$.

Note that the dynamic coupling between the platform and the robotic arm has not been considered yet. Therefore, the angular acceleration of the Chaser $\dot{\omega}_{CoM}^J \in \mathbb{R}^3$ (written in $\mathcal{F}_I$) does not include the component due to joints motion alone: $\dot{\theta}$.

In addition, it should be noted that only the linear terms (function of linear and angular acceleration) are considered in the Inertia Matrix $H_p$. The non linear terms (function of linear and angular velocities) are included in $C$ - remember equation (4.17).

The linearised expression for the total torque that actuates on the CoM of the system is given by

$$T_{CoM}^J = I_{tot}^J \dot{\omega}_{CoM}^J,$$  \quad (4.31)

where $I_{tot}^J \in \mathbb{R}^{3\times3}$ corresponds to the inertia tensor of the Chaser, and equals to the sum of the inertia of all substructures, which yields

$$I_{tot}^J = I_H^J + I_{Fuel}^J + I_{Ox}^J + I_{Arm}^J.$$  \quad (4.32)

The first three terms on the right hand side of equation (4.32) correspond to the inertia of the platform.
(hub, fuel and oxidizer tanks) and the last one to the inertia of the robotic arm.

Equation (4.17), is written w.r.t. to the inertial frame $\mathcal{F}_I$, hence one must find the Inertia of the Chaser w.r.t. this frame. However, it is easier to obtain the inertia tensor of a body w.r.t. to a body frame centred on its geometric center. Therefore, one shall translate and rotate the inertia tensor, written in the body frame, to the inertial frame. The rotation of a tensor was seen before and it is given by 4.22. To translate, we require the Parallel Axis Theorem. The translation from the center of mass C to a point A is expressed by

$$ I_A = I_C + m(r \times)(r \times)^T, $$

where $I_A, I_C \in \mathbb{R}^{3\times3}$ are the inertia tensor around point A and C, respectively; $m \in \mathbb{R}$ is the mass of the body and $(r \times) \in \mathbb{R}^{3\times3}$ is the skew symmetric of the vector from point C to point A, or vice versa.

Equation (4.17) is function of forces and torques that actuate on the platform and are written w.r.t. the acceleration of hub CoM not the Chaser CoM.

Translating forces and torques from the system CoM to the hub CoM, yields

$$ \begin{align*}
F_H &= F_{CoM}^q, \\
T_H &= T_{CoM}^q + r_{H\rightarrow CoM} \times F_{CoM}^q,
\end{align*} \tag{4.33} $$

where $F_H, F_{CoM} \in \mathbb{R}^3$ are forces that actuate on the hub CoM and on the Chaser CoM and $T_H \in \mathbb{R}^3$ is the torque actuating around the hub CoM.

From equations (4.29), (4.31) and (4.33), we obtain

$$ T_{tot}^q = I_{tot}^q \dot{\omega}_{CoM}^q + m_{tot} r_{CoM}^q \times a_H^q + m_{tot} r_{CoM}^q \times (\dot{\omega}_{CoM}^q \times r_{CoM}^q). \tag{4.34} $$

By replacing equation (4.29) in (4.28) and from equation (4.34), the expression for the platform inertia matrix $H_p$ is obtained and it is given by

$$ H_p = \left[ m_{tot} (r_{CoM}^q \times) \ I_{tot}^q - m_{tot} (r_{CoM}^q \times) (r_{CoM}^q \times) \right]. $$

### Coupling Inertia Matrix

The dynamic-coupling inertia matrix $H_{pm}$ expresses the contribution of the dynamic coupling between the manipulator and the platform [10].

Using the same principle as in equations (4.28)-(4.30), we obtain an expression for the force that the robotic arm produces on the platform $F_{pm} \in \mathbb{R}^3$, and it is given by

$$ F_{pm} = \sum_{i=1}^{n} m_i \ddot{q}_i z_i^q \times (r_{CoM_i}^q - r_i^q). \tag{4.35} $$

In fact, it is possible to relate equation (4.35) with the equation for the linear velocity Jacobian $J_v$ given...
by (4.10). Hence force $F_{pm}$ can be written as follows

$$F_{pm} = \sum_{i=1}^{n} m_i J_{v_i} \ddot{\theta}_i. \quad (4.36)$$

As before, it is required to transfer forces and torques from each segment of the manipulator to the hub CoM, just as in (4.33). Thus, one must set an analogous equation to (4.34). Term $F_{CoM}$, in system (4.33) is replaced by (4.36), the linear acceleration is null (joints only have angular acceleration), and term $I_{tot} \ddot{\omega}_{CoM}$ is replaced by $I_{Arm}^{\ddot{\theta}}$, which yields

$$I_{Arm}^{\ddot{\theta}} = \sum_{i=1}^{n} R_{I_i}R_{I_i}^{T} \ddot{\theta}_i z_i^3, \quad (4.37)$$

where $I_i \in \mathbb{R}^{3 \times 3}$ is the inertia tensor of segment $i$ obtained around its center of mass and written in the body frame $\mathcal{F}_i$.

From equations (4.33), (4.36), and (4.37), the expression for the torque that the manipulator produces on the platform is given by

$$T_{pm} = \sum_{i=1}^{n} [I_i J_{\omega_i} + m_i (r_{CoM_i}^3 - r_{CoM}^3) \times J_{v_i}] \ddot{\theta}_i, \quad (4.38)$$

where $I_i = R_i^3 R_i^T$ and $r_{CoM} \in \mathbb{R}^3$ is the center of mass of the whole spacecraft.

Therefore, the inertia matrix for the dynamics coupling is given by

$$H_{pm} = \begin{bmatrix} m_1 J_{v_1} & \cdots & m_n J_{v_n} \\ I_1 J_{\omega_1} + m_1 (r_{CoM_1}^3 - r_{CoM}^3) \times J_{v_1} & \cdots & I_n J_{\omega_n} + m_n (r_{CoM_n}^3 - r_{CoM}^3) \times J_{v_n} \end{bmatrix}. \quad (4.39)$$

### 4.3 Generalized Jacobian Matrix

Space robots, in on-orbit missions, use a robotic arm for tasks like de-orbiting of non malfunctioning satellites. These robots are divided into two main categories: free-floating and free-flying. In the former, the use of actuators at the base is avoided, i.e. the attitude and position of the base is controlled by the motion of the arm. Thus, the EE is able to track a desirable path while the satellite floats freely, which might be undesirable in some cases [6]. In the latter, the pose of the base is continuously controlled [5].

In the context of this project, a free-flying robot is considered, because keeping the same relative pose w.r.t. the Target is a requirement of the mission.

Integrating the upper set of equation (4.18), we obtain equation

$$\mathcal{L} = H_p \ddot{x}_H + H_{pm} \ddot{\theta}. \quad (4.39)$$

In [6], the Generalized Jacobian Matrix (GJM) is derived assuming conservation of momentum $L$. Gravitational forces and torques are not considered here. The former can be added later as a disturbance and the former is neglected. Therefore, we can assume conservation of momentum for free-floating robots.
The GJM is given by [6, 8]:

\[ J^* = J_m - J_p H_p^{-1} H_{pm} \]  

(4.40)

Now, one can write the kinematic relationship between the platform and the manipulator expressed by (4.19), only as a function of the joint rates, as follows

\[ \dot{x}_{EE} = J^* \dot{\theta}. \]

The IKP solution for redundant manipulators given by (4.16) can still be used, by replacing \( J^* \) by \( J \). By using GJM, one is eliminating the disturbances that the base causes on the manipulator motion. It should be noted that (4.40) holds true for free-floating robots but it is just an approximation for free-flying robots.
Chapter 5

Control Design Theory

In this chapter, we first analyse the controller architecture and how to achieve proper performance using feedback control. We shall give some guidelines to PID control as well as discuss theoretical design limitations imposed by bandwidth, open loop right half plane (RHP) poles.

In chapter 3, a model that represents the spacecraft with or without flexible appendages is presented. Chapter 4 discusses the inverse kinematics and dynamics of a spacecraft. Here, the knowledge gathered in the previous chapters culminates in control design; a clear understanding of how the information presented in the previous chapters relates to design control is presented.

5.1 Feedback Control Theory

Throughout this section, feedback control systems are considered. Figure 5.1 represents a simple feedback system, with a plant $G$, which represents the model considered, and some controller $K$. Inputs $r$, $d$, $n$ correspond to the reference to be tracked, disturbances and noise, respectively; whereas $y$ is the output, $y_m$ is the measured output, $u$ the control signal, and $e$ the tracking error. Feedback control is fundamentally important in the presence of model uncertainties ($\Delta$), signal disturbances ($d$) and an unstable plant [37].

![Feedback Control System Diagram]

Figure 5.1: One DOF negative feedback control system.

Model uncertainties are always present because a given plant $G$ is never an exact model of the real
system. As an example, the plant considered here is given by the TITOP model and another block that is basically composed by integrators and double integrators to obtain velocities and positions from accelerations (the latter being outputs from the TITOP block). This plant is represented in Figure 5.2. As known from Chapter 3, the TITOP assumes small angular rates, thus its quadratic terms are neglected. More detailed information about the plant considered in this project is provided in the next section 5.2.

An example of signal disturbances to this system is gravitational force. For more information about the system disturbances, the reader is referred to chapter 2. Sloshing is not considered an external disturbance because it is introduced in plant \( G \) using a spring-mass-damper model. However, the frequency \( \omega_n \) and damping \( \xi \) associated to the sloshing in the fuel and oxidizer tanks have associated uncertainties. The German aerospace center - Deutsches Zentrum für Luft und Raumfahrt (DLR) - studied this phenomenon for this mission, providing us the values for the sloshing mass, \( \omega_n \), \( \xi \), and the respective parametric uncertainties.

Unstable plants can only be stabilized by feedback control. Any linear system \( \dot{x} = Ax + Bu \) is stable if all its poles \( p_i \) are in the left-half plane (LHP), i.e. \( \text{Re}(p_i) < 0 \), or in other words, the eigenvalues of \( A \) are negative: \( \text{Re}[\lambda_i(A)] < 0 \) [37]. Another definition of stability is provided below.

**Definition 1. Stability.** A system is internally stable if none of its components contains hidden unstable modes and the injection of bounded external signals at any place in the system results in bounded outputs signals measured anywhere in the system [37].

From Figure 5.1, we obtain

\[
\begin{align*}
  y &= Gu + G_d d \\
  u &= K(r - y - n) \\
  e &= r - y
\end{align*}
\]

which yields a close loop response given by

\[
y = (I + GK)^{-1} GK r + (I + GK)^{-1} G_d d - (I + GK) GK n,
\]

Figure 5.2: Plant

An example of signal disturbances to this system is gravitational force. For more information about the system disturbances, the reader is referred to chapter 2. Sloshing is not considered an external disturbance because it is introduced in plant \( G \) using a spring-mass-damper model. However, the frequency \( \omega_n \) and damping \( \xi \) associated to the sloshing in the fuel and oxidizer tanks have associated uncertainties. The German aerospace center - Deutsches Zentrum für Luft und Raumfahrt (DLR) - studied this phenomenon for this mission, providing us the values for the sloshing mass, \( \omega_n \), \( \xi \), and the respective parametric uncertainties.

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  e &= r - y
\end{align*}
\]

which yields a close loop response given by

\[
y = (I + GK)^{-1} GK r + (I + GK)^{-1} G_d d - (I + GK) GK n,
\]
where $I$ is the identity matrix.

From equation (5.2), we define the Sensitivity function as

$$S = (I + GK)^{-1}$$

(5.3)

and the Complementary Sensitivity function as

$$T = (I + GK)^{-1}GK.$$  

(5.4)

The latter is called the complementary function of the former, because it follows the property

$$S + T = I.$$  

(5.5)

From equations (5.2) and (5.3), we can see that the Sensitivity function corresponds to the close loop transfer function from disturbances $d$ to the output $y$. Furthermore, from (5.2) and (5.4), one can note that the Complementary Sensitivity function is associated to the closed loop transfer function from the reference $r$ to the output $y$. It is also related to the transfer function from the perturbation input $n$ to the plant output $y$. Because $S$ and $T$ are complementary, there is an unavoidable trade off between disturbance attenuation and filtering measurement error.

For good reference tracking responses and good disturbance rejections, $S$ has to be small ($S \approx 0$) thus $T$ has to be big and close to the identity.

Another important relation from equations (5.2)-(5.4) follows

$$e = -Sr + SGd - Tn.$$  

(5.6)

Here, the close loop bandwidth $\omega_B$ is defined by the frequency where the magnitude of the Sensitivity function $|S|$ first crosses $\frac{1}{\sqrt{2}}$ from below. $|S|$ is below zero until $\omega = \omega_B$ and control is effective in improving performance. When $|S|$ is higher than 1 (i.e., 0 dB) control is no longer effective. A typical plot of the $|S|$ is represented in Figure 5.3.

![Figure 5.3: Plot of the magnitude of the Sensitivity function.](image)

A curious aspect of feedback control is its ability to "linearize" the behaviour of a system [37]. Feedback
can control the output \( y \) about an operating point. Hence, the system can remain in a linear region (e.g., keep the angular rate small) where linear models \( G \) and \( G_d \) are valid.

It should be reminded that the TITOP itself is not linear due to the rotation matrices in the blocks that represent the robotic arms segments. These rotation matrices are built with \( \sin \) and \( \cos \) functions of a time varying angle: \( \theta_i \). Since \( \sin \) and \( \cos \) are not linear, the TITOP model is also non linear, despite being based on linear equations for the dynamics and kinematics (small angular rate). It is possible to linearize the model using function `linearize` from MATLAB.

The linearization is done for some configuration \( \theta_0 \). The linearisation of the TITOP model is done for the same configuration \( \theta_0 \) and for zero joint velocities. Just as in [4], this linearisation holds for slow manoeuvres otherwise it should be revisited for more agile manoeuvres.

The linearization of \( H \) results in [4]

\[
H(\theta_0) \begin{bmatrix} \delta \dot{x}_H \\ \delta \dot{\omega}_H \\ \delta \dot{\theta} \end{bmatrix} = \begin{bmatrix} F_{ext} \\ T_{ext} \\ Cm \end{bmatrix}, \quad (5.7)
\]

where \( \delta \dot{x}_H \in \mathbb{R}^3 \), \( \delta \dot{\omega}_H \in \mathbb{R}^3 \), and \( \delta \dot{\theta} \in \mathbb{R}^n \) are variations around the nominal cases.

The major goals of feedback control, with unavoidable trade-offs in terms of the loop transfer function \( L = GK \) are listed below [37, 38].

- Stabilization of unstable plant: \( L \) large
- Reference Tracking: \( L \) large
- Good disturbance rejection: \( L \) large
- Good noise rejection: \( L \) small
- Small input signals: \( K \) and \( L \) small

Usually, the control signal \( u \) is required to be kept small in order to avoid fast changes in the system, hence avoiding high input energy.

We can have a better insight of the feedback trade-offs to be made, by carrying out frequency response analysis. Frequency response describes the system's response to sinusoids of varying frequencies. It is obtained from the Laplace transform by making \( s = j\omega \), where \( \omega \) is the sinusoid frequency [37].

The goals above are in different frequency ranges. Noise is usually of bigger importance at higher frequencies whereas reference tracking is more relevant at lower frequencies. Therefore, one can meet the objectives by using large loop gain \( |L| \) at low frequencies below crossover and small gain at high frequencies above crossover.

In Single-Input Single-Output (SISO) systems, frequency domain performance can be evaluated using bode plots and checking the gain and phase margins. The gain margin is a measure of how much loop gain can be increased before reaching close loop instability and it is expressed by [37]

\[
GM = \frac{1}{|L(j\omega_{180})|}, \quad (5.8)
\]
where $\omega_{180}$ is the frequency at which the phase margin is $180^\circ$.

Phase margin measures how much phase lag can be added to $L$ at $\omega_c$ before instability, and it is expressed by [37]

$$PM = \angle L(j\omega_c) + 180^\circ,$$  \hspace{1cm} (5.9)

where $\omega_c$ is called the crossover frequency and corresponds to the frequency where $|L(j\omega_c)| = 1$, or $GM = 0\, dB$.

Typically, $GM > 6\, dB$ and $PM > 30^\circ$ are required in order to obtain good performance. In fact, according to [37], $GM$ and $PM$ can be related to the maximum peaks of $S(M_S)$ and $T(M_T)$, by equations

$$GM \geq \frac{M_S}{M_S - 1}, \quad GM \geq 1 + \frac{1}{M_T},$$

$$PM \geq \frac{1}{M_S}, \quad PM \geq \frac{1}{M_T}.$$  

For good performance, the magnitude of the Sensitivity function must be considered.

In this project a Multiple-Inputs Multiple-Outputs (MIMO) system is considered. In a SISO system only scalars are considered. In contrast, in MIMO systems one must also consider directions for vectors and matrices. Directionality is provided by the Singular Value Decomposition (SVD). As in chapter 4 for the pseudo inverse of the Jacobian, matrix $G$ can be written as

$$G = U\Sigma V^T.$$  

Input and output directions are given by the column vector of $V$ and $U$, respectively. The elements of the diagonal matrix $\Sigma$ are called the singular values $\sigma_i$. Singular values are of quite importance, since the maximum (minimum) singular value $\bar{\sigma}$ ($\sigma$) corresponds to the maximum (minimum) gain for any input direction

$$\bar{\sigma}(G) = \max_{d \neq 0} \frac{\|Gd\|_2}{\|d\|_2},$$

$$\sigma(G) = \min_{d \neq 0} \frac{\|Gd\|_2}{\|d\|_2},$$  \hspace{1cm} (5.10)

and

where the vector-2 norm of some vector $v$ is given by

$$\|v(\omega)\|_2 = \sqrt{\sum_j |v_j(\omega)|^2}.\hspace{1cm} (5.12)$$

Before, we saw that the transfer function from the reference $r$ to the tracking error $e$ corresponds to $-S$ - see equation (5.6). In MIMO system, $S$ can also be used to measure the effectiveness of a feedback control system. Since directions are also important, one must consider the ratio $\frac{\|e(\omega)\|_2}{\|r(\omega)\|_2}$, which yields [37]

$$\sigma(S(j\omega)) \leq \frac{\|e(\omega)\|_2}{\|r(\omega)\|_2} \leq \bar{\sigma}(S(j\omega)),$$  \hspace{1cm} (5.13)
5.2 Analysis of the System

Before proceeding to the control design, one must analyse the plant in Figure 5.2. The block named TITOP is built with the model presented in chapter 3. The second block is built based on the set of equations given by

\[
\begin{align*}
\dot{x}_H(t_i) &= \int_{t_{i-1}}^{t_i} v_H dt + x_H(t_{i-1}) \\
\dot{v}_H(t_i) &= \int_{t_{i-1}}^{t_i} a_H dt + v_H(t_{i-1}) \\
\dot{q}_H(t_i) &= \int_{t_{i-1}}^{t_i} \dot{q}_H dt + q_H(t_{i-1}) \\
\dot{\omega}_H(t_i) &= \frac{1}{2} W(t_{i-1}) q_H(t_{i-1}) \\
\dot{\theta}(t_i) &= \int_{t_{i-1}}^{t_i} \dot{\theta} dt + \theta(t_{i-1}) \\
\dot{\dot{\theta}}(t_i) &= \int_{t_{i-1}}^{t_i} \ddot{\theta} dt + \dot{\theta}(t_{i-1})
\end{align*}
\]  

(5.14)

where \( W \in \mathbb{R}^{3 \times 3} \) is obtained by

\[
W = \begin{bmatrix} 0 & -\omega_H \\ \omega_H & (\omega_H \times) \end{bmatrix}.
\]

Note that in the simulation, discrete time is considered. Thus, \( t_i \) corresponds to the time at step \( i \) and \( t_{i-1} \) to the previous step.

The state vector of the system \( x \in \mathbb{R}^{31+n} \) is given by

\[
x = \begin{bmatrix} \eta & \dot{\eta} & x_H & \dot{x}_H & q_H & \omega_H & \theta & \dot{\theta} \end{bmatrix}^T,
\]

(5.15)

where \( \eta \in \mathbb{R}^3 \) corresponds to the displacement due to sloshing in the fuel and oxidizer tanks. Remember that each tank has 3 sloshing masses with different natural frequencies and damping, \( \dot{\eta} \in \mathbb{R}^3 \) is the time derivative of the displacement; \( x_H \in \mathbb{R}^3 \) is the inertial position of the hub CoM, and the \( \dot{x}_H \in \mathbb{R}^3 \) is the inertial linear velocity at that point, \( q_H \in \mathbb{R}^4 \) are the quaternions which represent the rotation from the body frame \( F_H \) to the inertial frame \( F_\mathbb{I} \), and \( \omega_H \in \mathbb{R}^3 \) is the hub angular velocity. It should be noted that all the elements of the platform (hub, fuel and oxidizer tanks, and the base of the robotic arm) have the same angular acceleration. Finally \( \theta, \dot{\theta} \in \mathbb{R}^n \) are the joint angles and joint rates, respectively.

The input vector \( u \in \mathbb{R}^{6+n} \) is given by

\[
u = \begin{bmatrix} F_{ext} & T_{ext} & Cm \end{bmatrix}^T,
\]

(5.16)

where \( F_{ext}, T_{ext} \in \mathbb{R}^3 \) correspond to the external forces and torques actuating on the hub. These inputs are written in the inertial frame, but measured in the body frame. The command torques at the joints are represented by \( Cm \in \mathbb{R}^n \).
The output $y \in \mathbb{R}^{31+n}$ is defined by

$$y = \begin{bmatrix} x_H & \dot{x}_H & q_H & \omega_H & \theta & \dot{\theta} \end{bmatrix}^T.$$  

(A5.17)

According to [37], state controllability of a system is defined by:

**Definition 2. State Controllability.** A dynamical system $\dot{x} = Ax + Bu$, or the pair $(A, B)$ is state controllable if there is an input $u$ such that $(x(t_f) = x_f$, for any initial state $x(0)$, any time $t_f > 0$, and any final state $x_f$. Otherwise, the system is uncontrollable [37].

In addition, the system $(A, B)$ is state controllable iff the Controllability Matrix defined by

$$C \equiv \begin{bmatrix} B & AB & A^2B & \ldots & A^{m-1}B \end{bmatrix}$$  

(A5.18)

has full rank $m$, where $m$ is the number of states.

Another definition for controllability is, for any eigenvalue $\lambda_i \in \mathbb{C}$ of $A$

$$\text{rank} \left[ (\lambda_i I - A) \ B \right] = m,$$  

(A5.19)

where $I$ is an identity matrix with the same size as $A$.

Controllability implies stabilizability.

**Definition 3. Stabilizability.** A system is stabilizable if all unstable modes are state controllable [37].

A system is stabilizable iff

$$\forall \lambda_i \in \mathbb{C}^+,$$

(A5.20)

Using the *linearize* function of MATLAB, we can study the controllability and stabilizability of the linearized chaser system. From the definitions above, the linearized system is **controllable** hence stabilizable.

Another important definition is state observability.

**Definition 4. State observability.** A dynamical system $\dot{x} = Ax + Bu$, or the pair $(A, C)$ is state observable if, for any time $t_f > 0$, the initial state $x(0) = x$ can be determined from the time history of the input $u(t)$ and the output $y(t)$ in the interval $[0, t_f]$. Otherwise, it is unobservable [37].

Controllability is to stabilizability, as observability is to detectability.

**Definition 5. Detectability.** A system is detectable if all unstable modes are state observable [37].

If a system is not detectable, there maybe a state which will eventually grow out of bounds, but it is impossible to observe this behaviour from the outputs $y$.  

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A system that is unstabilizable and undetectable has hidden unstable modes. It should be remarked that an unstable system can be stabilized by feedback control if there are no hidden modes.

The linear system considered is also observable, thus detectable.

In Figure 5.4, zeros and the poles of the system are represented. There are no right-half plane (RHP) zeros, however there is a LHP pole at $p \approx 1.4 \times 10^{-7}$. Thus, the system that represents the chaser is unstable. There are also poles at the origin, which correspond to the integrators in the plant.

![Pole-Zero Map](image)

Figure 5.4: Map of poles and zeros of the linear system considered.

It should be noted that the non linearities in the plant of the system are linearized by the inertia matrix $H$, which linearizes the TITOP block, and by feedback control by keeping the angular rate around an operating point, which corresponds to the tumbling rate of the target satellite. However, it does not necessarily mean that the system remains controllable, stabilizable, observable or detectable, since the system is prone to numerical errors during linearization and unstable modes may be introduced. Nevertheless, the analysis above provides us a prospect of how the system may behave.

### 5.3 Control Design

An accurate tracking and successful capture of the target satellite, with the robotic arm is desired. Following a smooth trajectory with low velocities and accelerations at the EE is of major importance due to the contact force between the EE and the Target at the Grappling Point (GP), where the impact could project the EE. In this framework contact forces will not be explored due to the complexity and extension of the problem. However, we must assure that the EE will reach the GP very slowly.

The processing capability of computers embedded in spacecrafts are of lower performance than the ones used on Earth. Hence, one must design a controller which requires low computational burden. An example is a simple PID controller, given by the transfer function

$$K = K_p + K_d s + \frac{K_i}{s},$$  \hspace{1cm} (5.21)

where $K_p$, $K_d$, and $K_i$ are the proportional, derivative gains, and integral gain respectively.

Controllers with derivative action are improper, i.e. the system tends to infinity when $w \rightarrow \infty$. To avoid improper behaviour, one needs to add a filter typically of the form \[37\]

$$F = \frac{1}{\epsilon \tau_d s + 1},$$  \hspace{1cm} (5.22)
where $\epsilon$ is a constant with a typically of 0.1 or less, and $\tau_D = \frac{1}{\omega_p}$.

In [7], a free floating space manipulator (no control at the platform) is considered and GJM is used to solve the inverse kinematics problem. A PD controller for the robotic arm to achieve point-to-point control or trajectory control is used, either in Cartesian space $x_{EE_d}$ or in Joint Space $\theta_d$. In contrast, in [4], a fixed structured $H_\infty$ control approach is used to synthesize both the control at the platform and at the manipulator. The regular $H_\infty$ optimal control problem finds a stabilizing controller $K$, which minimizes the $H_\infty$ norm of a function. The major problem of $H_\infty$ optimal control is that it provides a solution $K$, whose order is unknown beforehand and usually results in a high order controller, with inherent high computational burden. In contrast, in [4] and [39], a PD like structure is used and the respective gains are chosen based on minimizing an $H_\infty$ norm. More about both the regular and fixed $H_\infty$ control approaches is presented later.

The control design approach for the Chaser is based on a PID controller, whose gains are chosen either by making use of second order system characteristics (sections 5.3.1 and 5.3.2) or by applying a fixed structured $H_\infty$ control approach (5.4.3). This controller can be subdivided into two controllers: a PID controller for the platform and a PD controller ($K_i = 0$) for the robotic arm - see Figure 5.5. It should be noted that there is another important component of the controller: the inertia matrix $H$ defined in equation (4.17). This matrix is used to invert the dynamics and, thus, decoupling the dynamics of the system and linearizing it. Therefore, in theory, the plant to be controlled by the PID and PD controllers corresponds to the second block in Figure 5.2. In practice, it is not exactly the same due to numerical errors when computing the inverse dynamics matrix $H$.

The effect of the coupling between the platform and the robotic arm is studied in section 5.4. This coupling is represented by matrix $H_{pm}$, which is given by equation (4.39).

Note that the noise and disturbance inputs (from chapter 2) are not presented in Figure 5.5 for simplicity, but they are presented in Figure 1.3.

The actuators of the system are modelled as diagonal block of low-pass filters expressed by

![Figure 5.5: Coupled Controller. For simplicity, the outer-loop of the Manipulator’s controller is not represented. Blocks $K_{Plat}$ and $K_M$ correspond to the PID controller for the platform and the PD controller for the manipulator, respectively.](image)
\[ F = \frac{-1}{0.1s + 1}. \]  

(5.23)

### 5.3.1 Platform Control system

The accelerations \( a_H, \dot{\omega}_H, \) and, \( \ddot{\theta}_H \) and given in control law (4.18) are obtained using a PID control. In this section, only the accelerations at the platform (hub + base + fuel tank + oxidizer tank) are considered. Remember that \( \ddot{x}_P \in \mathbb{R}^6 \) represents linear and angular accelerations.

The close loop transfer function, with plant \( G = \frac{1}{s^2} \) and a PD controller given by (5.21), is defined as

\[ T = \frac{k_ds + k_p}{s^2 + k_ds + k_p}. \]  

(5.24)

A second order system is characterized by transfer function

\[ T = \frac{2\omega_n \xi s + \omega_n^2}{s^2 + 2\omega_n \xi s + \omega_n^2}. \]  

(5.25)

The natural frequency \( \omega_n \) is the frequency at which the system would oscillate if the damping ratio \( \xi \) would be zero.

Comparing (5.24) to (5.25), one obtains equalities

\[
\begin{align*}
    k_p &= \omega_n^2 \\
    k_d &= 2\omega_n \xi
\end{align*}
\]  

(5.26)

To make the close loop stable one must ensure that there are no RHP poles, thus constants \( k_p \) and \( k_d \) must be positive.

The resolved linear acceleration is written as [40]

\[ a_P = \ddot{r}_{H_d} + k_{d_p} \Delta \dot{r}_H + k_{p_p} \Delta r_H, \]  

(5.27)

where \( k_{d_p} \) and \( k_{p_p} \) are positive PD gains and \( r_{Pd} \in \mathbb{R}^3 \) is the desired inertial relative position between hub CoM and the Target CoM. Thus, it corresponds to the reference of the control system; \( \Delta r_H \in \mathbb{R}^3 \) is the position error, which is defined by

\[ \Delta r_H = r_{H_d} - r_H. \]  

(5.28)

where \( r_H \in \mathbb{R}^3 \) is the inertial relative position of the CoM of the hub (main substructure of the Chaser), w.r.t. the Target. Equations (5.27) and (5.28) yield a close loop dynamic behaviour given by

\[ \Delta \dot{r}_H + k_{d_p} \Delta \dot{r}_H + k_{p_p} \Delta r_H = 0. \]

In order to reduce residual error, integral action with small gain \( k_i \) is introduced. Hence the control law is given by equation (5.21). Figure 5.6 shows the PID controller using the resolved linear acceleration.
from equation (5.27). A constant term, representing the centrifugal acceleration was also introduced as a disturbance to the system, since this term is still quite significant for the system dynamics.

\[
\begin{align*}
\frac{1}{s} &+ K_p &+ K_i &+ K_d \\
& & &+ a_{centr}
\end{align*}
\]

Figure 5.6: PID controller using the hub inertial position error.

Euler angles have two major disadvantages: they have singularities and are less accurate then unit quaternions when used to integrate incremental changes in attitude over time [41]. Quaternion \( q \) is a four dimensional vector

\[
q = \begin{bmatrix} q_0 & q_1 & q_2 & q_3 \end{bmatrix}^T.
\]  

(5.29)

The first element \( q_0 \) corresponds to the scalar part and \( \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix}^T \) corresponds to the vector part representing the rotation axis [42].

The resolved angular acceleration is given by [41]

\[
\alpha_P = \dot{\omega}_d + k_{dP_a} \Delta \omega_H + k_{pP_a} \epsilon,
\]  

(5.30)

where \( k_{dP_a} \) and \( k_{pP_a} \) are feedback gains, and \( \Delta \omega_H \in \mathbb{R}^3 \) is the angular rate error defined by

\[
\Delta \omega_H = \omega_{Hd} - \omega_H,
\]

where \( \omega_{Hd} \in \mathbb{R}^3 \) is the desired angular rate at the hub, written in the inertial frame and \( \omega_H \in \mathbb{R}^3 \) is the inertial angular rate of the hub. The orientation error \( \epsilon \in \mathbb{R}^3 \) is obtained with the vector part of the quaternions and it can be written as

\[
\epsilon = R \epsilon_{\mathcal{F}_T \rightarrow \mathcal{F}_H},
\]  

(5.31)

where \( R \in \mathbb{R}^{3 \times 3} \) is the DCM from the hub body frame \( \mathcal{F}_H \) to the inertial frame \( \mathcal{F}_I \); \( \epsilon_{\mathcal{F}_T \rightarrow \mathcal{F}_H} \in \mathbb{R}^3 \) corresponds to the vector part of the quaternion representing the rotation from the Target’s frame \( \mathcal{F}_T \) to \( \mathcal{F}_H \). This quaternion \( q_{\mathcal{F}_T \rightarrow \mathcal{F}_H} \in \mathbb{R}^4 \) is obtained with quaternion multiplication given by

\[
q_{\mathcal{F}_T \rightarrow \mathcal{F}_H} = q_{\mathcal{F}_H \rightarrow \mathcal{F}_T} \otimes q_{\mathcal{F}_T \rightarrow \mathcal{F}_H},
\]

where \( \otimes \) stands for quaternion multiplication and \( q^* \) is the quaternion conjugate of \( q \).

The closed-loop dynamic behaviour corresponds to equation

\[
\Delta \dot{\omega}_H + k_{dP_a} \Delta \omega_H + k_{pP_a} \epsilon = 0.
\]  

(5.32)
Figure 5.7 shows the PID controller using the resolved angular acceleration in 5.30, with an integral action to eliminate offset.

![Figure 5.7: PID controller using the hub attitude error.](image)

However, equation (5.31) is non-linear which makes (5.32) non-linear as well. In [40], a Lyapunov candidate function is defined, in order to ascertain closed-loop stability. The candidate function is defined by

$$V = k_{p_{pa}} \left[ (\eta - 1)^2 + \epsilon^T \epsilon \right] + \frac{1}{2} \Delta \omega_H^T \omega_H. \quad (5.33)$$

where $\eta \in \mathbb{R}$ corresponds to the scalar part of quaternion $q_{F_F \rightarrow F_H}$ and $k_{p_{pa}}$ is positive constant.

The time derivative of $V$ in (5.33), along the trajectories of system (5.32) is given by

$$\dot{V} = -k_{d_{pa}} \Delta \omega_H^T \omega_H. \quad (5.34)$$

where $\dot{V}$ is negative semi-definite if $k_{d_{pa}}$ is positive, therefore the system is asymptotically stable and converges to two equilibria [40]

$$\begin{cases} 
\text{Equilibrium 1: } \{ \eta = -1, \epsilon = 0, \Delta \omega_H = 0 \} \\
\text{Equilibrium 2: } \{ \eta = 1, \epsilon = 0, \Delta \omega_H = 0 \} 
\end{cases} \quad (5.35)$$

The first equilibrium is unstable. The reader is referred to [40], where a detailed explanation of this instability is provided. The system must converge to the second equilibrium which implies that successful tracking is achieved.

As mentioned above, the feedback gains corresponding to the PD controller are related to the natural frequency of the system $\omega_n$ and to the damping ratio $\xi$.

On the other hand, for second order systems, the damping ratio is related to the overshoot by equation [43]

$$\xi = \frac{-\ln \left( \frac{\text{OS}}{100} \right)}{\sqrt{\pi^2 + \ln \left( \frac{\text{OS}}{100} \right)^2}} \quad (5.36)$$

where $\text{OS}$ corresponds to the percent overshoot.

A system with high natural frequency corresponds to a faster system.

One can define the settling time ($t_s$) of a system as the

**Definition 6.** Settling time. Time for the step response $y(t)$ to reach and stay within 2% of the
steady state value [43].

For second order system responses, the settling time can be given by [43]

\[ t_s = -\frac{\ln\left(0.02\sqrt{1-\xi^2}\right)}{\xi \omega_n} \]  \hspace{1cm} (5.37)

Hence, for a typical overshoot of 5% one obtains a damping ratio of \( \xi = 0.69 \). It is required that the system moves very slowly so that the linear model holds. Hence, for a frequency of \( \omega_n = 0.3 \text{rad/s} \) and a damping ratio of \( \xi = 0.7 \) the settling time is around 20s.

The feedback gains are, thus \( k_{p_p} = 0.09, k_{d_p} = 0.42, k_{p_R} = 0.09, \) and \( k_{d_R} = 0.42 \). It should be noted that this choice of gains is just a starting point.

In 5.22, \( \epsilon \) was set to 0.05.

5.3.2 Manipulator Control system

The manipulator controller is more complex than the platform controller. It is composed by an outer-loop and an inner-loop, as represented in Figure 5.9. The goal is for the EE to track and capture the GP of the Target. To do so, one has to compute the velocity that the EE requires, in order to reach the GP’s pose.

The computation of the velocity is based on the pose error between the EE and the GP. This velocity is then mapped into joint rates, using the Generalized Jacobian Matrix (GJM). It should be reminded that the GJM is obtained with the current configuration of the robotic arm joints.

Velocity computation and mapping to joint rates compose the outer-loop, shown in blue in Figure 5.9. Joint rates \( \dot{\theta}_d \in \mathbb{R}^n \) are integrated to obtain joint angles \( \theta_d \in \mathbb{R}^n \). These quantities are now references to an inner control loop which computes the feedback angular acceleration \( \ddot{\theta} \) with an independent PD control, given by

\[ u = K_{p_M} \theta_e + K_{d_M} \dot{\theta}_e, \]  \hspace{1cm} (5.38)

where \( K_{p_M}, K_{d_M} \in \mathbb{R}^{n \times n} \) are positive diagonal matrices, whose diagonal elements correspond to proportional and derivative gains, respectively. \( \theta_e \in \mathbb{R}^n \) is an array with the joints angle error. This error is computed with the current configuration \( \theta \in \mathbb{R}^n \)

\[ \theta_e = \theta_d - \theta \]
and \( \dot{\theta}_e \) is the joint rates error obtained with the current joint rate \( \dot{\theta} \) and yields

\[
\dot{\theta}_e = \dot{\theta}_d - \dot{\theta}.
\]

\[r_p\]
\[r_M\]
\[IKP\]
\[\theta_d, \dot{\theta}_d\]
\[\text{Controller} \]
\[\alpha, \omega\]
\[K_{\text{Plat}}, P_{\text{mat}}\]
\[\dot{\theta}\]
\[H\]
\[T_{\text{ext}}, P_{\text{mat}}\]
\[C_m\]
\[\theta\]
\[\omega\]
\[\text{EE}\]
\[v_{\text{EE}}, \omega_{\text{EE}}\]
\[r_H, \omega_H, \theta_H\]

Figure 5.9: Simplified representation of the feedback controller. Both the inner and outer loops of the Manipulator’s controller are represented.

As shown in [30], this PD controller should achieve asymptotic tracking of the desired joint positions.

One can choose the elements of \( K_{P,M} \) and \( K_{d,M} \) based on the characteristics of a second order system given above

\[
\begin{align*}
K_{P,M} &= \text{diag} \{ \omega_1^2, \ldots, \omega_n^2 \} \\
K_{d,M} &= \text{diag} \{ 2\omega_1 \xi_1, \ldots, 2\omega_n \xi_n \}
\end{align*}
\]

(5.39)

The PD controller, where the angular acceleration \( \ddot{\theta} \) (input of the \( H \) block) is resolved, is represented in Figure 5.8.

Integral action was not introduced because references \( \theta_d \) and \( \dot{\theta}_d \) keep changing until the position and velocity error at the end effector is around zero.

As a starting point, one may choose the same gains \( k_{P,i,M} \) and \( k_{d,i,M} \) for each joint \( i \in [1, \ldots, n] \). In [??], \( \omega_i \) corresponds to 20 times the GP’s angular rate, which corresponds to a frequency of \( \approx 0.87 \text{rad/s} \). For a smooth trajectory, it is required a high damping ratio, e.g. \( \xi_i = 0.7 \). It should be noted that a frequency of \( \omega_i = 0.88 \text{rad/s} \) and a slow arm motion are **not** incompatible, since the velocity at the EE is obtained in the inner-loop, and it can be tuned via \( \lambda \) in the IKP solution given by (4.16) and by using individual weights for the linear and angular EE’s velocity: \( X_{v,EE} \) and \( X_{\omega,EE} \). Usually, in tracking it is desirable that the attitude goal is achieved before the position. One can give higher priority to the attitude by making \( X_{\omega,EE} \) higher than \( X_{v,EE} \).

Figure 5.10 shows how much the magnitude of \( S \) changes with \( \lambda \). For higher values of \( \lambda \), \( |S| \) tends to decrease, i.e. a slow motion of the EE is more beneficial to the performance of the system. It should also be noted that after \( \lambda = 0 \) there is a deep decrease in the magnitude of \( S \). This result is compatible to what was presented in chapter 4, since by making use of redundancy in the manipulator with equation (4.16), one is prioritising slow motion at the EE.
However, a high value of $\lambda$ also implicates high settling time $t_s$. Hence, there must be a trade-off between the velocity of the system and disturbance rejection.

In addition, the behaviour of $|S|$ when facing a change PD gains, is studied. In Figure 5.11a, PD gains, corresponding to joint 1 ($k_{p1,M}$ and $k_{d1,M}$), are changed using equation (5.39) and by setting different values to frequency $\omega_1$.

The same is shown in Figures 5.11b, 5.11c, and 5.11d, except now for joints 2, 4, and 7, respectively. Lastly, Figure 5.11e shows the behaviour of $|S|$ by varying $k_{p_i,M}$ and $k_{d_i,M}$ simultaneously for every joint $i \in \{1, \ldots, n\}$.

From Figures 5.11a-5.11d, we can see that if one changes gains $k_{p_i}$ and $k_{d_i}$ corresponding to some joint $i$, the neighbouring joints will be more affected by that change than further joints. It is also noticeable that until around the value of $\omega$ set for the remaining joints, in this case 0.88 rad/s, $|S|$ decreases as $\omega_i$ increases; after that, $|S|$ increases again.

Surprisingly, if one changes $\omega_i$ simultaneously for every joint, the magnitude of $S$ decreases with $\omega_i$. However, due to the waterbed effect of the Sensitivity function, by decreasing its magnitude around some frequency, it increases in another part of the spectrum. In this case, it decreases around the bandwidth frequency (frequency at which $|S|$ crosses 0dB from below) and may increase at lower frequencies where it is important to keep $|S|$ low. Note that, by increasing the PD gains, the control signal $u$ will have higher magnitude, which might be incompatible with the actuators of the system.

5.4 Controller Structures

In 5.3.1 and 5.3.2, a PID controller was defined for the platform and a PD controller for the manipulator inner loop. These gains can be gathered in a single diagonal matrix. This diagonality means that coupling between the platform and the manipulator and the contributions between joints were not considered. Decentralized control uses diagonal or block diagonal controllers. According to [37], this approach works well if plant $G$ is close to diagonal.

Plant $G$ is not diagonal due to the coupling between the platform and the robotic arm, and due to sloshing in the Fuel Tank. However, by introducing the inertia matrix $H$ (given in (4.17)) in our controller, we are decoupling the dynamics of the system. If the sloshing would not be considered and apart from possible
Figure 5.11: Magnitude of Sensitivity function versus variable ω_i (from equation (5.39)) at different joints.

numerical errors, $H$ matrix would correspond to the inverse of the TITOP model represented above. Hence, we would obtain an ideal shaped plant $G_s$, equivalent to the second block in Figure 5.2.

5.4.1 Coupled Control

Inertia matrix $H$ introduces coupling between the platform and the manipulator, which is represented by $H_{pm}$. In addition, it introduces the effect that each joint produces on the other joints $H_m$ and the kinematic relation between angular and linear accelerations at the hub $H_p$. Figure 5.5 represents the inner control system with the $H$ matrix. To this control approach, we call coupled control.

5.4.2 Independent Control

In contrast to coupled control, independent control does not consider coupling inertia $H_{pm}$. Hence, platform motion is considered to be unaffected by the motion of the joints in the robotic arm (and vice versa). Hence, the dynamic decoupling is not as perfect. This controller is called independent controller and is represented in Figure 5.12.
Figures 5.13 and 5.14 represent the system’s response to a step change in reference $\theta_d$ using coupled and independent control, respectively. In these tests, the Chaser is considered to be fixed in the inertial frame, hence the angular rate is null and there is no sloshing. With these settings it is clear how the system reacts to a damped step change in the manipulator reference. The initial joint angles are $\theta_0 = \begin{bmatrix} 0 & 45 & 0 & -45 & 0 & 0 & 0 \end{bmatrix}$ deg and the reference corresponds to a damped step whose final setting is $\theta = \begin{bmatrix} 0 & 45 & 30 & -45 & 0 & 0 & 0 \end{bmatrix}$ deg.

Using coupled control, a change in the manipulator reference reflects a change in the control inputs corresponding to the manipulator - $C_m$ - and the platform - $F_H$ and $T_H$. On the other hand, using independent control, only a change in $C_m$ is noticeable. In the first case, there is good dynamic decoupling, where only joint 3 move. In the second case, it is noticeable that the decoupling is not perfect. In addition to joint 3, other joints also move to compensate for the coupling between the platform and the manipulator, which is considered in $G$ but not in $H$.

Independent control should only be used if the coupling effect between the platform and the manipulator is not important. This is the case when the platform has much higher mass and inertia than the manipulator. However, once the robotic arm captures some target, with similar or higher mass and inertia to the platform, the coupling effect between platform, and the manipulator and the Target, might be too high to reasonably apply independent control.

Therefore, the goal, now, is to understand if independent control holds when the Chaser has captured some target and the manipulator is retracted. Similarly to the previous tests, the platform is fixed in the inertial frame. The characteristics of the target are presented in Table 5.1.

<table>
<thead>
<tr>
<th>Target’s Characteristics</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I_{xx}$ [kg/m$^2$]</td>
</tr>
<tr>
<td>$I_{xy}$ [kg/m$^2$]</td>
</tr>
<tr>
<td>$I_{yy}$ [kg/m$^2$]</td>
</tr>
<tr>
<td>$I_{zz}$ [kg/m$^2$]</td>
</tr>
<tr>
<td>Mass [kg]</td>
</tr>
</tbody>
</table>

Table 5.1: Target’s characteristics.
A simple test, and with only a few changes in the dynamic model used above, can be made. Instead of adding an additional substructure to the model, one can increase the mass and inertia of the manipulator last segment, according to the settings in 5.1. The results using coupled and independent control are presented in Figures 5.15 and 5.16, respectively. In Figures 5.15e and 5.16e, reference angles are plotted in full line and the joint angles are plotted in dashed line.

From the results shown above, it is clear that independent control must not be used when coupling between the platform and the manipulator cannot be neglected, i.e. Coupling matrix $H_{bm}$ is of the same or higher order as the platform inertia matrix $H_b$.

Using coupled control, the pose of the hub CoM remains the same and only the 4th joint moves (as commanded). Hence, we can conclude that the diagonalization and the linearization of the model is successful. It should be reminded that in these tests the non linear and time variant TITOP model was used dually with the non linear and time varying inertial matrix $H$. In contrast, using independent control where the diagonalization and thus the linearization of the system is not achieved, the pose of the hub CoM and the joint angles change drastically.
Figure 5.14: Response of the Chaser to external commands, considering independent dynamics between the platform and the robotic arm.

5.4.3 Structured $H_\infty$ Control Problem

Structured $H_\infty$ is a new approach to control design which allows the engineer to directly and efficiently tune control architectures with a given structure. These architectures could be PID controllers, gains, filters, etc. As the name suggests it is based on $H_\infty$ control theory, thus a small introduction to this optimal control approach is presented below.

An $H_\infty$ is an optimal control algorithm, which minimizes the $H_\infty$ norm of the lower linear fractional transformation

$$z = F_l(P, K)w,$$

where $P$ is the extended Plant and $K$ is the controller; $w$ correspond to the normalized exogenous inputs of $P$, i.e. reference commands, measurement noise, and disturbances. The normalized exogenous outputs are represented by $z$ and include performance error and the force and torque commands. The $H_\infty$ norm of some transfer function $f(s)$ is given by

$$\|f(s)\|_\infty \triangleq \max_{\omega} |f(j\omega)| \Leftrightarrow \max_{\omega} \sigma(f(j\omega)),$$
and yields

$$\|F_l(P, K)\|_\infty = \max_{\omega} \sigma (F_l(P, K)(j\omega)).$$  \hspace{1cm} (5.40)$$

If we wish to obtain good disturbance-rejection performance, good noise attenuation and maintain robust stability in the presence of multiplicative perturbations, then it is required to keep the sensitivity $S$ and the complementary sensitivity $T$ small; for good reference tracking $T$ should be big. However, due to relationship (5.5), it is not possible to keep both small or big at the same frequency. Nevertheless, as stated above these requirements are more demanding at different frequencies. Hence, one can solve a weighted $H_\infty$ by making

$$F_l(P, K) = \begin{bmatrix} W_1 S \\ W_2 T \end{bmatrix},$$  \hspace{1cm} (5.41)$$

where both $W_1$ and $W_2$ are frequency dependant weighting matrices; $W_1$ is required to be small in the required bandwidth of the system and equal to 1 for higher frequencies; $W_2$ should be approximately 1 in the frequency range of the system and roll off for higher frequencies [37].

For more complex systems, one might need to give more information about exogenous signals, in addition to the signals to be minimized. This approach is called signal-based $H_\infty$ control and it is shown in Figure 5.15: Response of the Chaser to external commands, considering coupling between the platform and the robotic arm.
Figure 5.16: Response of the Chaser to external commands, considering coupling between the platform and the robotic arm.

Figure 5.17 [37].

However, solutions to the $H_{\infty}$ problem tend to be of higher order than real industrial control problems can handle. In the face of vast requirements that should be met versus computational burden, control engineers felt the need to create a new approach to control design. With Structured $H_{\infty}$ one can write the requirements as weighted frequency dependent functions as a normal $H_{\infty}$ problem, and solve the minimization problem providing a pre defined shape of controller $K$ [4, 39, 44], which in this case is two
PID and one PD controllers and two weighting gains for the EE’s linear and angular velocity: $X_{v_{EE}}$ and $X_{\omega_{EE}}$, respectively.

Any linear control system can be written in the standardized representation of Figure 5.18, where we have a Linear Time Invariant (LTI) plant $P$, exogenous inputs and outputs - $w$ and $z$ respectively - and some controller $C$. All reference signals and disturbances are gathered in $w$ and performance related outputs in $z$ [39, 44].

In Structured $H_\infty$ control, the augmented plant $P$ includes, not only plant $G$ but also all the remaining fixed blocks, e.g. blocks EE and IKP in Figure 5.9 (excluding gain $X_{v_{EE}}$ and $X_{\omega_{EE}}$, included in block IKM). $C$ corresponds to a diagonal matrix with all tunable control elements, which may be repeated in matrix $C$. It should be noted that any interconnection of LFT models is a LFT model. Since the Standard LFT model is a LFT model, if we have some rational function with tunable elements, the structure of this function is absorbed into $P$ and the tunable elements are included in diagonal matrix $C$. In [39, 44], provides an example using a low-pass filter $F = \frac{a}{s+a}$. $C$ matrix corresponds to

$$C = \text{diag}(a,a)$$

and $P$ is given by

$$P = \begin{bmatrix}
0 & 0 & 1 \\
1/s & -1/s & 0 \\
1/s & -1/s & 0
\end{bmatrix}.$$  

Using MATLAB, we can easily obtain the Standard Form of some system by setting tunable elements with function `realp` and by using function `slTuner` to obtain the Standard Form of a Simulink model [4].

In contrast to the $H_\infty$ control, in Structured $H_\infty$ control one can constrain more than one close loop transfer functions, by repeating the controller multiple times in the Standard Form [39, 44].

To specify the requirements, a signal based control approach was used, where the goal can be written in the form of

$$\|F_1(P,C)\|_\infty \leq \gamma, \quad (5.42)$$

where $\gamma$ is equal to 1 or slightly higher than one for suboptimal control problems.

Gains $W_r$, $W_d$ and $W_n$ may be constant or dynamics and describe their relative importance and frequency content. $W_d$ corresponds to a constant diagonal matrix to normalize the input $d$, since it un-
known its importance in the frequency spectrum. \( W_r \) is low pass filters; \( W_{rP} \) corresponds to the platform reference weight with magnitude at low frequency of \( r_{Pd} \), where \( r_{Pd} \) is the desired distance between target and the chaser, thus, normalizing input \( r \); the analogous process is done for the EE reference \( W_{rP} \). The gain at the attitude is 1 since the quaternions (either for the platform or the EE) are already normalized. \( W_n \) corresponds to high pass filters (one for each plant output).

\( W_{ref} \) is used to provide information about the desired close loop response and corresponds to a second order system written as

\[
W_{ref} = \frac{1}{s^2 + 2\omega_n \xi + \omega_n^2},
\]

where \( \omega_n \) is the desired natural frequency of the system and \( \xi \) the desired damping ratio.

\( W_{refP} \) was specified for the close loop response of the platform pose, with an opening at the output \( \theta \); the natural frequency response was set to \( \omega_n = 0.3 \) and the damping ratio to \( \xi = 0.7 \).

We want the EE to move slowly and according to the ESA specifications in [2], the capture should take 5 minutes, i.e., 300s. Using equation (5.37), for the settling time as a function of the natural frequency and damping ratio, and by setting \( t_s = 250s \) and \( \xi = 0.7 \), one obtains \( \omega_n = 0.024rad/s \).

Therefore, \( W_{refEE} \), which corresponds to the close loop response of the EE’s position and written in the form of 5.43, is set with \( \omega_n = 0.024 \) rad/s, and \( \xi = 0.7 \). Since, it is desirable that the EE’s attitude is settled before the EE’s pose, the settling time is slightly lower: \( t_s = 150s \), hence obtaining \( \omega_n = 0.041rad/s \), for \( W_{refEE} \).

\( W_e \) provides information about the desired frequency content of the error \( e \) and it is desired that \( |W_e S| < 1 \). Hence, \( W_e \) shall correspond to a low pass filter and it is given by [37]

\[
W_e = \frac{s + \omega}{s + A \omega}.
\]

Typically, it is required that the maximum peak of \(|S|\) is less than 2 (6 dB), thus \( M = 2 \). According to [37], large values of the maximum peak of \(|S|\) and \(|T|\), i.e., larger than 4, indicates poor performance and robustness.

\( W_{eP} \) is used for the platform position error and \( A = \frac{0.01}{r_{Pd}} \) to force the position error lower than 0.01 m at low frequencies. \( W_{eP} \) corresponds to the gain used for the platform attitude error and \( A = 5 \times pi/180 \). The frequency \( \omega \) for both cases equals 0.3 rad/s.

Regarding the EE, we have \( A = 0.005 m \) and \( \omega = 0.024 rad/s \), and \( A = 0.5 pi/180 \) and \( \omega = 0.041 rad/s \) for the position and attitude, respectively.

Finally, \( W_u \) is used to keep the control signals \( u \) below some value. \( W_{Fu} \), \( W_{Tu} \) and \( W_{Cmu} \) are set for the control inputs \( F_{ext} \), \( F_{ext} \) and \( F_{ext} \), respectively.

\( W_u \) is a high pass filter given by

\[
W_u = \frac{s + \omega}{s + A \omega},
\]

where \( \omega \) is the crossover frequency and \( A \) the magnitude for low frequencies.

\( W_{Fu} \) was set with \( \omega = 0.3 rad/s \), and \( A = 100 N \); \( W_{Tu} \) with \( \omega = 0.3 rad/s \), and \( A = 90 Nm \); the weight for the control inputs was set with \( \omega = 1rad/s \), and \( A = 20 Nm \) (the nominal back-drive torque of the joint.
The requirements can now be written in the form of transfer functions between exogenous inputs and output. These transfer functions are expressed by

\[ z_r = T_r w_r, \]
\[ z_{rEE} = T_{rEE} w_{rEE}, \]
\[ z_q = T_q w_q, \]
\[ z_{qEE} = T_{qEE} w_{qEE}, \]

where subscripts \( r_P \) and \( q_P \) correspond to the platform position and attitude, respectively, and subscripts \( r_{EE} \) and \( q_{EE} \) to the EE’s position and attitude.

We can then use function \textit{systune}, from MATLAB, and specify the requirements as frequency-weighted gain constraint. The resulting gains in \( C \) are listed in Table 5.2.

<table>
<thead>
<tr>
<th>Resulting Elements of ( C )</th>
<th>Platform</th>
<th>End Effector</th>
</tr>
</thead>
<tbody>
<tr>
<td>( k_{p_r} )</td>
<td>0.421</td>
<td>0.013</td>
</tr>
<tr>
<td>( k_{i_r} )</td>
<td>5.1 \times 10^{-5}</td>
<td>1.69</td>
</tr>
<tr>
<td>( k_{d_r} )</td>
<td>0.085</td>
<td>1.69</td>
</tr>
<tr>
<td>( k_{p_q} )</td>
<td>3.8 \times 10^{-4}</td>
<td>1.69</td>
</tr>
<tr>
<td>( k_{i_q} )</td>
<td>0.432</td>
<td>1.69</td>
</tr>
<tr>
<td>( k_{d_q} )</td>
<td></td>
<td>1.69</td>
</tr>
<tr>
<td>( X_{v_{EE}} )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( k_{p_1} )</td>
<td></td>
<td>0.237</td>
</tr>
<tr>
<td>( k_{p_2} )</td>
<td></td>
<td>2.08</td>
</tr>
<tr>
<td>( k_{p_3} )</td>
<td></td>
<td>2.08</td>
</tr>
<tr>
<td>( k_{p_4} )</td>
<td></td>
<td>2.08</td>
</tr>
<tr>
<td>( k_{p_5} )</td>
<td></td>
<td>2.08</td>
</tr>
<tr>
<td>( k_{p_6} )</td>
<td></td>
<td>2.08</td>
</tr>
<tr>
<td>( k_{p_7} )</td>
<td></td>
<td>2.08</td>
</tr>
<tr>
<td>( X_{\omega_{EE}} )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( k_{d_1} )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( k_{d_2} )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( k_{d_3} )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( k_{d_4} )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( k_{d_5} )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( k_{d_6} )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( k_{d_7} )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Function \textit{systune} also outputs the information about the best achieved values of goals. And it is expressed through vector \( fSoft \) with elements less than 1 when the requirement is fulfilled. The elements of \( fSoft \) are presented in Table 5.3.

<table>
<thead>
<tr>
<th>Best achieved soft constraint values</th>
</tr>
</thead>
<tbody>
<tr>
<td>( T_r )</td>
</tr>
<tr>
<td>1.59</td>
</tr>
</tbody>
</table>

From Table 5.3, we can see that the results for the platform are quite positive, since \( T_q \) is below 1. The result \( T_r \) is higher than 1, but the computed gains might still result in good stability and performance for the desired frequency range. Although the \( fSoft \) elements for the EE are not close to 1, a deeper analysis should be done or order to study stability and robustness for the desired frequency range.
5.5 Robust Control Theory

According to [37], in order to get acceptable performance, the presence of uncertainty requires the use of feedback control. The sensitivity reduction with respect to uncertainty is achieved with high gain feedback but for any real system we have a crossover frequency range where the loop gain has to drop below 1 and the presence of uncertainty in this frequency range may result in poor performance or even instability.

5.5.1 Uncertainties

Uncertainties are unavoidable in real systems, since any mathematical model is only an approximation of a real system. These approximations include unmodeled dynamics, which are usually more important at higher frequencies, system-parameter variations due to environmental changes, and neglected nonlinearities (e.g., neglect centrifugal acceleration) [45].

In robust control design, one must specify the uncertainties that the system should be robust against, so that the uncertain model captures the essential features of the real system. In addition, the uncertainty class should be chosen taking into account whether it leads to a tractable solution to the robust control problem.

One can enlarge the class of uncertainties in some system in order to enlarge the tractability of the robust control problem. Hence, increasing the mathematical model similarity to the real system. Nonetheless, there should be a trade-off between tractability of the robustness analysis and conservatism of the uncertain system model [46]. There are several forms of describing uncertainties, but they are typically a magnitude bounded quantity. If \( \Delta(s) \) is an uncertain transfer function, it yields:

\[
|\Delta(j\omega)| \leq \mu \quad \forall \omega
\]

Uncertainties can be divided into two classes: unstructured and parametric uncertainties, which are introduced in the following sections.

Unstructured uncertainties

Dynamic perturbations that occur in different part of a system can be lumped into a single uncertainty block \( \Delta \). A common example are unmodeled dynamics usually described in the high-frequency domain. These types of uncertainties are called unstructured and may include time delays, parasitic coupling and other nonlinearities [45].

Neglected and unmodelled dynamics uncertainty is somewhat less precise than parametric uncertainties and thus more difficult to quantify but it appears that the frequency domain is particularly well suited for this class [37].

In the context of this thesis, second order terms were neglected. The angular velocity affects the natural frequency of the system as studied in [47]. Since, we are considering a slow-motion of the system, with an inherent low angular acceleration, these types of uncertainties are not introduced in the control
system. With the advantage of not increasing its complexity. However, a future analysis of the system should include these types of uncertainties in order to ensure that the system operates properly under a different set of assumptions.

**Parametric uncertainties**

Model uncertainties are not only caused by unmodeled system dynamics, but also by ill-defined system characteristics. Such perturbations may be represented by variations of certain system parameters over some possible value ranges (complex or real), and affect the low-frequency range performance [45]. A parametric uncertainty model may be deceiving since it could provide a detailed and accurate description, while the underlying assumptions about the model and the parameters may be much less exact. In addition, if using real perturbations, numerical problems may arise. According to [37], the latter could be resolved by introducing complex perturbations instead. Although, possible plants that are not present in the original set may be introduced. In MATLAB, `complexify` method can be used to replace uncertain real parameters by uncertain complex ones.

The parametric uncertainties, introduced in this thesis, are the mass and inertia of each element represented in the system (i.e. main body, the base of the robotic arm, the 7 robotic arm segments, and the fuel and oxidizer tanks). Additionally, the natural frequency and damping of each part of each tank (the fuel and oxidizer tanks are subdivided into 3 parts with different characteristics - mass, inertia, natural frequency and damping). The properties with the highest uncertainty correspond to the dampings of the mass-spring-damper system of the tanks (50% range over their nominal value), followed by their mass and natural frequency (20% uncertainty). This reflects the little knowledge in characterizing the behaviour of the fluid motion inside the tanks. These uncertain real parameters are inserted in the system using the `ureal` function from MATLAB, and then switched to complex parameters, using the `complexify` method.

Figure 5.19, shows a Linear Fractional Transformation (LFT) of the system with the uncertainty block $\Delta$, pulled out of the respective subsystem $i$ of the TITOP model. Each of these uncertainty blocks are diagonal matrices with repeated members. The reader is referred to review the

![Figure 5.19: Simplified LFT representation taking into account uncertainties.](image)

After including the uncertainties in the system the another solution to the $H_\infty$ was found. The gains of
control blocks to be tuned are presented in Table 5.4.

Table 5.4: Resulting Elements of $C$, using Fixed Structured $H_{\infty}$ control and with disturbances included in the system.

<table>
<thead>
<tr>
<th>Platform</th>
<th>$k_{pr}$</th>
<th>$k_{ir}$</th>
<th>$k_{dr}$</th>
<th>$k_{p1}$</th>
<th>$k_{i1}$</th>
<th>$k_{d1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.45</td>
<td>$1.4 \times 10^{-9}$</td>
<td>0.11</td>
<td>1.98</td>
<td>0.04</td>
<td>11.41</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>End Effector</th>
<th>$X_{v,EE}$</th>
<th>$k_{p1}$</th>
<th>$k_{p2}$</th>
<th>$k_{p3}$</th>
<th>$k_{p4}$</th>
<th>$k_{p5}$</th>
<th>$k_{p6}$</th>
<th>$k_{p7}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.06</td>
<td>0.53</td>
<td>3.21</td>
<td>0.43</td>
<td>1.97</td>
<td>2.06</td>
<td>13.57</td>
<td>6.3</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>End Effector</th>
<th>$X_{\omega,EE}$</th>
<th>$k_{d1}$</th>
<th>$k_{d2}$</th>
<th>$k_{d3}$</th>
<th>$k_{d4}$</th>
<th>$k_{d5}$</th>
<th>$k_{d6}$</th>
<th>$k_{d7}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.65</td>
<td>0.32</td>
<td>0.98</td>
<td>0.42</td>
<td>5.78</td>
<td>5.02</td>
<td>5.59</td>
<td>8.12</td>
</tr>
</tbody>
</table>
Chapter 6

Control Analysis

In this chapter, time and frequency domain analysis are performed to understand if the system obtained in chapter 5, with gains according to table 5.4, is stable and robust towards disturbances and uncertainties of the system.

These controllers were tuned according to chapter 5. Tuning is necessary in order to reach certain requirements. In this chapter, we are going to analyse the solution tuned with structured $H_{\infty}$ control approach, where uncertain parameters were introduced in the system.

6.1 Problem Description

Once we define a design model, the primary goal of the feedback design problem is nominal stability. In SISO systems, the Nyquist criterion is used; this criterion can be extended to MIMO cases by requiring that the number of encirclements of $\det(I + GK)$ (instead of $1 + GK$, in SISO cases) equals to the number of unstable open loop poles. Clockwise encirclements are considered positive and anticlockwise, negative.

If nominal stability is established, the evaluation of nominal performance is the next concern. Finally, robust stability and performance to uncertainties $\Delta$ of the system, such as sloshing mass and sloshing frequency, are analysed.

Throughout section 6.2, coupled control is considered.

In this section, a thorough analysis of the tuned controller is presented. This analysis is done considering the gains selected in sections 5.3.1 and 5.3.2.

The system to be studied is a multi body system with flexible appendages, where sloshing is also considered. Thus, to study stability and performance, cuts at different locations of the loop were made.

This is a common engineering approach used in the industry [48], providing the engineer a more intuitive understanding of complex system's behaviour towards different inputs and parameters. Figure 6.1 represents loop break at the plant output $r_H$, with a red cross.

To study nominal stability at the output of the plant, a cut at that output of the plant is made. We obtain open loop transfer functions $L_o = GK$ by linearising the system with a break at the loop, as depicted
Figure 6.1: Coupled Controller. For simplicity, the outer-loop of the Manipulator’s controller is not represented.

In Figure 6.1. The linearisation is obtained with function \textit{linearize} from MATLAB, which outputs \( L_o \). It should be noted that since only one loop break is done at each time, some cross interconnections are not considered in \( L_o \).

If cuts are made at the input of the plant we obtain the open loop transfer function \( L_i = KG \).

After computing \( L_o \), it was noticeable that the respective poles in the open RHP are very close to the origin, i.e. of order below \( 10^{-9} \). These RHP poles could come from internal instabilities due to dynamic decoupling or just numerical errors from linearisation and minimal realization. Hence, the Nyquist criteria could not provide useful information regarding stability of the system.

Nevertheless, one can test if by feeding a bounded input to the system, the outputs remain bounded as well.

6.2 Robust Solution

Throughout this section, the control system obtained using the \( H_\infty \) structured approach and uncertain disturbances is considered. Similarly to the nominal solution in C, a time and frequency domains analysis are carried out. This section is called “Robust Solution”, however it only means that uncertainties were introduced in the system prior to finding the solution to the structured \( H_\infty \) control problem. It does not necessarily mean that the solution found is more robust than the nominal solution.

Next, a series of Figures, showing the system’s transient response to a step input at the reference of the end effector (EE) position, is presented. The Manipulator tries to follow a certain trajectory, while keeping EE attitude and Platform’s pose. Figure 6.2 shows the trajectory of the EE overlapped with the trajectory of the grappling point (GP), at the Target, and the point where capture was achieved. Here, capture is accomplished when the position error and attitude error, at the EE, are below 5\textit{mm} and 0.5\textit{deg}, respectively.
6.2.1 Time Domain Analysis

Transient responses of both the Platform and the Manipulator are represented in Figures 6.3 and 6.4, respectively. This simulation was subjected to random noise and disturbances. The disturbances added to the simulation are quite pessimistic comparing to environmental disturbances expected in chapter 2. Before the capture phase, stabilization phase takes place, where the Platform's pose is to be stabilized. In this simulation, it is assumed that the stabilization phase was not perfect and hence, at \( t = 0 \) s, the initial position and attitude errors, at the Platform, are not zero. In addition, there is a small initial angular rate error - see Figures 6.3.

The attitude error in 6.3c at the hub has a smoother transition when compared to the response in Figure C.1c. Since gains \( X_{v_{EE}} \) and \( X_{\omega_{EE}} \) are larger comparatively to the respective gains in the nominal solution, the robotic arm response is much faster: having the EE reached the GP at \( t = 1 \text{ min}40 \). However, it should be noted that a faster response means that the command torques at the robotic arm are also larger. Similarly to the nominal results, the command torque the highest at joints associated to the robotic arm segments with the highest inertia - segments 2 and 4. Here, the maximum command torque achieved at these joints is of 37.36Nm and 13.8Nm, for joints 2 and 4 respectively, whereas for the previous solution the maximum torques were 2.7Nm and 1.0Nm. Note that a command torque of 37.36Nm is still compatible to the actuators capabilities at the joints. According to the requirements these joints shall be able to provide a repeated peak torque of 176Nm and momentary peak torque of 340Nm.

Figures 6.5, 6.6, and 6.7 show the results to the system step responses to disturbances at \( F_{\text{ext}} \) for the first two figures and \( Cm_1 \) and \( Cm_4 \) at the third. Multiple step responses are plotted for \( r_H \), \( q_H \), and \( \theta_1 \) and \( \theta_4 \), considering different value for the uncertain parameters presented in Chapter 5. In the first two Figures, the hub is able to reach the desired pose, for multiple values of uncertain parameters. However, the 4th quaternion element shows the highest variance. On the other hand, both joints show a consistent behaviour to a step at input \( Cm_1 \) and \( Cm_4 \), respectively.
6.2.2 Frequency Domain Analysis

Here, an analysis of the control system is provided, in the frequency domain. Figure 6.8 shows singular values of S at different outputs of the plant, as a function of frequency. The
Figure 6.5: Step response of the Hub CoM position to disturbances at the input $F_{ext}$, considering different values for the uncertain parameters presented in chapter 5.

Figure 6.6: Step response of the Hub attitude (quaternion) to disturbances at the input $F_{ext}$, considering different values for the uncertain parameters presented in chapter 5.

Figure 6.7: Step response of the 1st and 4th joints to disturbances at inputs $C_{m1}$ and $C_{m4}$, respectively. Different values for the uncertain parameters, presented in chapter 5, are considered.
results for the position and attitude at the hub are quite similar to the ones obtained above. However, the waterbed effect is noticeable in Figure 6.8c, for the joint angles. By decreasing the singular values magnitude at lower frequencies, an increase of the magnitude is noticeable around $\omega = 1\text{Hz}$. For higher frequencies the singular values magnitude remain below $W_c$, as desirable.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{singular_values}
\caption{Singular values of the Sensitivity function $\sigma(S)$ at different outputs of the plant.}
\end{figure}

Figure 6.8 shows the singular values of $S$ at the inputs of the plant. For the torque and force at the hub, the singular values of the Sensitivity function are below 0 for frequencies between 0.01 Hz and 10Hz, resulting in good attenuation of disturbances. In contrast, the singular values sensitivity function at the joints command torque is quite above 6dB (the maximum magnitude allowed). Therefore, the feedback control introduced is not sufficient to attenuate disturbances at the input.

The singular values for the complementary sensitivity function are represented on Figure 6.10.

In order to study robust stability, the stability margin using method \texttt{robstab} from MATLAB. Computing this margin is equivalent to computing the structured singular value for some block structure. The structured singular $\mu$ is a function which provides a generalization of the singular value $\bar{\sigma}$. The goal is to find the factor $k$ which makes the system unstable. The smallest $k$ which makes $I - M \Delta$ singular corresponds to $1/\mu$, where the $M \Delta$ is defined according to Figure 6.11 [37].

Figure 6.12, shows the upper bound of the stability margin as a function of frequency for the robust solution presented here, for the nominal solution with gains from Table 5.2 and for the manual solution, whose gains were selected according to a second order system, with $w = 0.3\text{rad/s}$ and $\psi = 0.7$ for the platform controller and $w = 0.88\text{rad/s}$ and $\psi = 0.7$ for the manipulator controller. More plots regarding the manual solution are presented in appendix C. The lower bound of the robust stability margin presented very similar results to the upper bound.

The three solutions present very similar results and resulted in stable solutions. Robust performance means that the performance goal is satisfied, for all possible plants in the uncer-
Figure 6.9: Singular values of the Sensitivity function $\sigma(S)$ at the plant’s inputs.

(a) Singular Values measured at the Hub’s position $r_H$. (b) Singular Values measured at the Hub’s attitude $q_H$. (c) Singular Values measured at the joint angles $\theta$. (d) Singular Values measured at the EE’s position $r_{EE}$. (e) Singular Values measured at the EE’s attitude $q_{EE}$.

Figure 6.10: Singular values of the Complementary Sensitivity function $\sigma(T)$ at different locations of the system.
Figure 6.11: MΔ structure for robust stability analysis.

Figure 6.12: Stability margin

tainty set even the the worst case plan [37]. Thus, in order to study robust performance, the variance was calculated. First, for different values of uncertain parameters - Figure 6.13. Then, for different values of the target tumbling rate ($\omega_y = \{0, \cdots, 2\omega_{Ty}\}$rad/s, where $\omega_{Ty}$ is the actual $y$ component of the target tumbling rate) - see Figure 6.14. The results consider the trajectory in Figure 6.2.

Changes in the tumbling rate $\omega$ produce higher variance in the results than varying the uncertain parameters. The hub position is the most affected by tumbling rate, having a variance of $0.02$m. It was also noticeable that the higher the initial angular rate error between the target and the chaser, the higher the variance. This tests reflects the importance of the angular rate in the dynamics of the system and, as a consequence, the importance of the effect of the centrifugal acceleration. Note that constant $a_{centr}$ from Figure, was also updated for each value of $\omega$. For the other outputs the variance was not so significant.
Figure 6.13: Variance of different outputs, for different values of the uncertain parameters.
Figure 6.14: Variance of different outputs, for different values of the target tumbling rate.
Chapter 7

Conclusions

Due to the escalating accumulation of debris in LEO, its removal has become a concern. Debris cause damage to currently operating satellites in space, and, in turn, produce more debris. In order to mitigate this problem, ESA is studying mission to capture a large sized un-operational satellite, tumbling at a constant rate of $\sim 0.05 \text{ rad/s}^2$. One of the phases of this mission is the capture itself, using a robotic arm attached to a small satellite. The main goal is for the robotic arm end effector to reach the target satellite Launch Adapter Ring, whilst keeping the pose of the chaser satellite.

The first step was to understand the dynamics and kinematics of the system $\text{chaser satellite + robotic arm}$, and analyse eventual simplifications to introduce in the mathematical model of the system. The TITOP introduced in [15] allows us to model complex system, which may including flexible appendages. This is an incredibly versatile solution, since it is quite adaptable to different systems, where neglecting second order terms viable. This model is then introduced in the control system, representing the system dynamics.

Because the robotic arm joint angles are not constant, the mathematical model obtained is non-linear. The Inverse Kinematic Problem (IKP) and the Jacobian matrix were then introduced. The IKP is concerned with what joint angles are required to reach the desired pose and the Jacobian is used to map the spacial velocity of end effector into joint rates. The inverse dynamic problem followed next and it is used to compute the commands to apply on the chaser satellite and on the robotic arm. This is done by building the $H$ inertia matrix. This matrix is also used in the control system to linearise the system.

On chapter 5, control design theory is introduced as well as the advantages of feedback control. Different design approached are then studied, as well as the influence of coupled control vs. decoupled control. Since every mathematical model is only an approximation of the real system, uncertainties should be introduced. The structured $H_\infty$ control problem provides a solution, if possible, to the control problem using a pre-defined structure of the control system. This approach computes $H_\infty$ norm to reach the established requirements written as weighted transfer functions.

The solutions of the structured $H_\infty$ control approach - one with uncertainties introduced in the system and another without - were then analysed and compared in chapter 6. It was expected the the solution which included the disturbances resulted in better robustness. However, because the structured $H_\infty$
control problem is non-convex the requirements defined were not achieved at all frequencies. Also there may exist multiple solutions to the control problem, using this approach. The randomness in finding the solution to the control problem and the fact that the requirements were not fully fulfilled in both situations (with or without disturbances), prevents us of providing a fair comparison between both solutions. Although, in the time domain the system was able to reach the goal and keep within bounds, in frequency domain did not reach the requirements for all frequencies, specially when analysing the outputs/inputs corresponding to the robotic arm.

7.1 Future Work

An important part of tracking the target with the robotic arm is the guidance introduced in the system. For that the IKP was used together with the damped least squares method to avoid numerical errors. In addition to this method, one can make use of the redundancy of the robotic arm (7 joints in a 6DoF problem) in order to avoid colliding with the chaser satellite and the robotic arm. An interesting study would be to provide different trajectories to the robotic arm and analyse how it behaves. Next, the results from the simulation could be compared to the results of the real robotic arm following the same trajectory, and using the controller designed.

A different approach to control design could be done: solving the $H_{\infty}$ problem followed by a order reduction in order to be compatible with the technology used in space.

Further analysis in time and frequency domain should also be carried on, e.g. make Monte Carlo simulation to analyse the outputs for different configurations of the uncertainties.
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Appendix A

Denavit Hartenberg Representation

The Denavit Hartenberg (DH) Representation is often used to define the geometry of a robotic manipulator. This is an efficient systematic approach to choose the reference frame of each joint of a robotic manipulator. The DH convention is defined by 4 parameters: link length \(a\), link twist \(\alpha\), joint offset \(d\), and joint angle \(\theta\) \([49]\). \(d_i\) and \(\theta_i\) \((i \in \{0, \ldots, n\}\) where \(n\) is the total number of joints) are variable parameters if joint \(i\) is prismatic or revolute, respectively; the other parameters are always constant. An Homogeneous Transformation matrix \(T_{i-1}^i\) is built with these 4 parameters. This transformation matrix represents the Rotation and the Translation from frame \(F_i\) to frame \(F_{i-1}\).

For revolute or prismatic joints there is only one variable parameter. Using this convention the Transformation matrix only changes when that parameter changes. This is where the computational efficiency of this representation relies.

In fact, there are two DH conventions. The DH parameters mentioned above are measured along different frames and the order at which the rotations and translations are performed is also different. Hence, resulting in two different transformation matrices.

A.1 Modified DH convention

This is the convention used along this project.

A revolute or prismatic joint \(i\) rotates or slides along \(z_i\), respectively. Axis \(x_i\) is defined along the common normal between \(z_{i-1}\) and \(z_i\). The \(y\) axis is the axis that completes a right-hand reference system. The DH parameters can be defined as \([50]\):

- \(a_{i-1}\): distance from \(z_{i-1}\) to \(z_i\), along \(x_{i-1}\)
- \(\alpha_{i-1}\): angle from \(z_{i-1}\) to \(z_i\), about \(x_{i-1}\)
- \(d_i\): distance from \(x_{i-1}\) to \(x_i\), along \(z_i\)
- \(\theta_i\): angle from \(x_{i-1}\) to \(x_i\), about \(z_i\)
It is not possible to represent any Homogeneous Transformation matrix only using four parameters [49]. In the Modified DH convention, only Transformation Matrices that hold the form (A.1) are considered. In addition two constraints must be followed in formulating the frames:

- Axis $x_i$ must intersect with axis $z_{i+1}$
- Axis $x_i$ must be perpendicular to axis $z_{i+1}$

In [31], the steps of attaching frames using this convention are enumerated.

Next, a small introduction to Homogeneous Transformation matrices follows. Any Homogeneous Transformation matrix $T_{ij}^j \in \mathbb{R}^{4\times4}$ is given by:

$$T_{ij}^j = \begin{bmatrix} R_{ij}^j & O_{ij}^j \\ 0 & 1 \end{bmatrix}$$

where $R_{ij}^j \in \mathbb{R}^{3\times3}$ and $O_{ij}^j \in \mathbb{R}^3$ express the orientation and position from frame $F_i$ to frame $F_j$, respectively.

In order to transform some vector $r^i = [x, y, z]^T$ from $F_i$ to $F_j$, one must add a row with value one to perform the translation represented by $O_{ij}^j$. Hence, the transformation is given by:

$$\begin{bmatrix} r^j \\ 1 \end{bmatrix} = T_{ij}^j \begin{bmatrix} r^i \\ 1 \end{bmatrix}$$

where $r^j \in \mathbb{R}^3$ is the position vector after the rotation and translation expressed in the transformation matrix $T_{ij}^j$. It should be noted that first the vector is rotated and then translated.

In the modified DH convention, the order of rotations and translations is from $F_i$ to $F_{i-1}$:

$$T_{i-1}^{i-1} = R_x(\alpha_{i-1})O_x(a_{i-1})R_z(\theta_i)O_z(d_i)$$  \hspace{1cm} (A.1)

where the Rotation matrices $R_x$ and $R_z$, respectively, are given by:

$$R_x(\alpha_{i-1}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha_{i-1}) & -\sin(\alpha_{i-1}) \\ 0 & \sin(\alpha_{i-1}) & \cos(\alpha_{i-1}) \end{bmatrix} \hspace{1cm} R_z(\theta_i) = \begin{bmatrix} \cos(\theta_i) & -\sin(\theta_i) & 0 \\ \sin(\theta_i) & \cos(\theta_i) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ \hspace{1cm} (A.3)

and the Translation matrices $O_x$ and $O_z$ are, respectively, expressed by:

$$O_x(a_{i-1}) = \begin{bmatrix} a_{i-1} \\ 0 \\ 0 \end{bmatrix} \hspace{1cm} O_z(d_i) = \begin{bmatrix} 0 \\ 0 \\ d_i \end{bmatrix}$$ \hspace{1cm} (A.4)

Finally, we obtain the Homogeneous Transformation for the modified DH-convention matrix $T_{i-1}^{i-1}$:

$$T_{i-1}^{i-1} = \begin{bmatrix} \cos(\theta_i) & -\sin(\theta_i) & 0 & a_{i-1} \\ \sin(\theta_i) & \cos(\theta_i)\cos(\alpha_{i-1}) & -\sin(\alpha_{i-1}) & -d_i\sin(\alpha_{i-1}) \\ \sin(\theta_i) & \cos(\theta_i)\sin(\alpha_{i-1}) & \cos(\alpha_{i-1}) & d_i\cos(\alpha_{i-1}) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$ \hspace{1cm} (A.6)
As an example, to perform the transformation from the end effector of a robotic arm with 7 segments to the base, one has to perform the following step:

\[ T_7^0 = T_1^0 T_2^1 \cdots T_6^7 \]  

Equation (A.7) holds irrespectively of the convention used.

### A.2 Classical DH convention

In contrast to the modified convention, a revolute or prismatic joint \( i \) rotates or slides along \( z_{i-1} \), respectively. Axis \( x_i \) is defined along the common normal between \( z_{i-1} \) and \( z_i \). The intersection point between axis \( z_{i-1} \) and \( z_i \) define the origin of frame \( F_i, O_i \). The \( y_i \) completes the right-hand reference frame [49].

According to [49], the parameters of the classical DH Representation are defined as:

- \( a_i \): distance along \( x_i \) from \( O_i \) to the intersection of the \( x_i \) and \( z_{i-1} \)
- \( \alpha_i \): angle between \( z_{i-1} \) and \( z_i \), about \( x_i \)
- \( d_i \): distance along \( z_{i-1} \) and \( z_i \) from \( O_i \) to the intersection of axis \( x_i \) and \( z_{i-1} \)
- \( \theta_i \): angle from \( x_{i-1} \) to \( x_i \), about \( z_{i-1} \)

Similarly to the modified convention, there are two constrains when formulating the frames [49]:

1. Axis \( x_i \) intersects axis \( z_{i-1} \)
2. Axis \( x_i \) is perpendicular to axis \( z_{i-1} \)

In the classical DH convention, the Homogeneous Transformations matrix follows the order of rotation and translation given below:

\[
T_{i-1}^i = R_z(\theta_i)O_z(d_i)O_x(a_i)R_x(\theta_i)
\]

where matrices \( R_x, R_z, O_x \) and \( O_z \) are given by equations (A.2), (A.3), (A.4), and (A.5), respectively. In [49], the procedure to establish the DH frames is summarized.

Finally, the transformation matrix is expressed by:

\[
T_{i-1}^i = \begin{bmatrix}
\cos(\theta_i) & -\sin(\theta_i)\cos(\alpha_i) & \sin(\theta_i)\sin(\alpha_i) & a_i\cos(\theta_i) \\
\sin(\theta_i) & \cos(\theta_i)\cos(\alpha_i) & -\cos(\theta_i)\sin(\alpha_i) & a_i\sin(\theta_i) \\
0 & \sin(\alpha_i) & \cos(\alpha_i) & d_i \\
0 & 0 & 0 & 1
\end{bmatrix}
\]  

(A.8)
Appendix B

TITOP Validation Extra Plots

Figure B.1: Angular acceleration and angle at the first joint. Response to a constant torque of $\vec{T}_{A/P} = 5\hat{z}$ applied at Body 1. The results of the model developed in Simscape are overlapped with the results from the TITOP model.

Figure B.2: Angular acceleration and angle at the second joint. Response to a constant torque of $\vec{T}_{A/P} = 5\hat{z}$ applied at Body 1. The results of the model developed in Simscape are overlapped with the results from the TITOP model.
(a) Linear acceleration about the $x$ axis: $a_x$.

(b) Linear acceleration about the $y$ axis: $a_y$.

(c) Linear acceleration about the $z$ axis: $a_z$.

Figure B.3: Linear acceleration at the Hub’s CoM. Response to a command torque $C_m = 10N\cdot m$, on the 1st joint. The plots in full line correspond to the Simscape Model and the plots in dashed line to the TITOP model.

(a) Angular acceleration around the $x$ axis: $\dot{\omega}_x$.

(b) Angular acceleration around the $y$ axis: $\dot{\omega}_y$.

(c) Angular acceleration around the $z$ axis: $\dot{\omega}_z$.

Figure B.4: Angular acceleration at the Hub. Response to a command torque $C_m = 10N\cdot m$ acting on the 1st joint. The plots in full line correspond to the Simscape Model and the plots in dashed line to the TITOP model.
Appendix C

Nominal Controller

In this chapter, the results obtained with the controller, tuned with Structured $H_{\infty}$ control, are presented. The sensor noise parameters are the same as used for the baseline solution. As explained in section 6.1, it was not possible to prove nominal stability resorting to the Nyquist criteria. Hence, Time Domain analysis is used to understand if the outputs of the system stay bounded to bounded inputs. Throughout section 6.2, coupled control is considered.

C.1 Time Domain Results

The transient response of the system to the same perturbations as in 6.2 is presented in Figure C.1. It is clear, that in the presence of disturbances and noise, the pose of the hub CoM and of the EE remain bounded. Hence, we can assume nominal stability.

The solution obtained from Structured $H_{\infty}$ results in a slow motion at the EE, reaching the GP with a position error of 5mm, only at $t = 7\text{min}30\text{s}$. By reducing the velocity at the EE, the sensitivity to disturbances is lower. There is a never ending trade-off between having a faster response and higher sensitivity to disturbances of the system.

C.2 Frequency Domain Analysis

Frequency analysis of the system is presented below. Figure C.3 show the singular values corresponding to Sensitivity function at the Plant outputs: position of the Hub’s CoM ($r_H$), Hub’s attitude ($q_H$), and joint angles ($\theta$). The dashed line corresponds to the weighted frequency function $W_e$, described in 5.4.3. We can see that the response of the joints angle is much faster than for the outputs $r_H$ and $q_H$.

Since $r_H$ passes over $W_e$, the bandwidth is smaller and the system is slower, which does not reflect important drawbacks in time domain. The attitude is noticeable above the requirement. This result might be due to problems resulting from linearisation when analysing frequency domain.

The singular values of the Sensitivity function at the plant inputs $F_H$ and $T_H$ keep below $W_e$, between frequencies 0.01rad/s and 0.1rad/s. The a singular value of $S$ at $Cm$ is around 40dB, which means that
the performance is not as good for some input direction $Cm$. The Complementary Sensitivity, measured at different points of the system, is shown in Figure C.5. The end effector follows the reference for very small frequencies $\sim 10^{-2}\text{rad/s}$, which reflects its slow
velocity in time domain. In contrast, the joint angles have a crossover frequency at $\sim 1\text{rad/s}$. It is actually desired that the joint angles quickly track its reference since the latter changes as a result of the Trajectory Generator block.

Next the robustness of the system is tested for different uncertain parameters. Note that these parameters were changed one parameter at each time.

The maximum Sensitivity function at the plant output $r_H$ is plotted as a function of uncertain parameters, in Figure C.6. The uncertain parameters considered here are the sloshing damping, sloshing mass, and Hub’s mass. The damping has the highest uncertainty, varying from $\xi = 0.01$ to $\xi = 0.1$. Sloshing mass
has an uncertainty of 20% and Hub’s mass of 5%. Nevertheless, \(|S|\) only decreases 0.03dB, which is irrelevant. The uncertainty of the mass of the Hub and sloshing mass also produce insignificant changes at the maximum Sensitivity function. These uncertain parameters produce even less noticeable results for the other Plant outputs.

The maximum Sensitivity function is plotted as a function of mass of different Segments, in Figure C.7. Here, we consider only the first and last Segments, and the Segment with the highest inertia: 4th Segment; all of them with a mass uncertainty of 5%. |S| change is not significant for any of these parameters. Still, the middle joint has the biggest decrease and increase in |S|: 0.15dB increase for |S| at the plant input \(C_{m4}\) and 0.005dB decrease at the plant output \(\theta_4\).

We can see that the uncertain parameters do not degrade the robustness of the system. It should be noted, that by setting uncertain parameters at a time, one is not taking into consideration possible harmful parameters configurations. However, it gives a close understanding of what kind of uncertainties might be the most harmful to the system. \(\mu\) analysis may be used as an extension of this analysis.
Figure C.6: Maximum Sensitivity at the output plant $r_H$ as a function of uncertain parameters.

(a) Maximum Sensitivity function at the Plant output $r_H$, for different values of sloshing damping.  

(b) Maximum Sensitivity function at the Plant output $r_H$, for different values of sloshing mass.

(c) Maximum Sensitivity function at the Plant output $r_H$, for different values of Hub's mass.

Figure C.7: Maximum Sensitivity as a function of some uncertain Segments' mass. The plots correspond to the 1st, the 4th and the 7th Segments.