Improved Generation of Correlated Gaussian Random Variates by Inverse DFT

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**Important notice**

To the best of the authors knowledge, the results in this report are correct and accurate. However and due to its preliminary status, some errors may still subsist. Permission to use the results in this report is granted provided the results are duly acknowledged and referenced.
Abstract

In this report the problem of generating a stationary band-limited Gaussian random vector with arbitrary complex autocorrelation by the inverse discrete Fourier transform (IDFT) algorithm is considered. Instead of using the classical frequency mask (FM), determined from samples of the (band-limited) target power spectral density (PSD) of the process, a new FM is obtained by matching the autocorrelation obtained with the IDFT algorithm to a desired arbitrary autocorrelation. Example results presented show that the new FM is able to significantly increase the autocorrelation accuracy of the generated process at no additional online computational cost.

Keywords: Gaussian random vector, inverse discrete Fourier transform, fading channel simulation.
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1 Introduction

The generation of a vector of Gaussian random variates with an arbitrary covariance matrix is a problem of considerable practical interest in many engineering fields, particularly when system simulation is required. Of particular importance is the field of digital communications where the effect of additive noise and fading on system performance is usually assessed by means of computer simulation. A number of general methods have been proposed for the generation of a discrete-time, $N$-sample, stationary or nonstationary Gaussian vector with an arbitrary covariance matrix. The most important of these general methods, which provide random variates with an \textit{exact} target correlation matrix, are the \textit{rotation} \cite{1}, the \textit{conditional} \cite{2,3} and the \textit{triangularization} \cite{3,4} methods. The latter method seems to have been proposed independently by Marsaglia \cite{4}, as a simplification to the method proposed by Stein & Storer \cite{5} involving an eigenvalue decomposition of the covariance matrix, and also later, by Scheuer & Stoller in \cite{3}. These three methods have roughly the same $O(N^2)$ real-time complexity but the triangularization method appears to be the less computationally demanding \cite{1}. These methods are thus practical if only a small number of samples needs to be generated and no correlation error may be tolerated. If the process is stationary then the covariance matrix is Toeplitz and it suffices to specify the process autocorrelation\textsuperscript{1}. In this case, approximate yet simpler methods have been proposed for the generation of Gaussian serially correlated random variates, namely the \textit{filtering} method \cite{6–8} and the \textit{inverse discrete Fourier transform} (IDFT) method introduced by Smith \cite{9}. While neither method provides variates with an exact target autocorrelation, the error is controllable and may be reduced to acceptable levels by adequate algorithm design. It is worth noting that a seldom cited, yet very interesting precursor to \cite{9} is the method proposed by Zondek in \cite{10} although the use of the IDFT is not suggested. The IDFT is a frequency-domain filtering algorithm that uses a frequency mask (FM) obtained from samples of the desired power spectral density (PSD) at a set of $N$ discrete frequencies. The computation of the IDFT is done efficiently in $O(N \log_2 N)$ operations using the fast Fourier transform (FFT) algorithm and, for a given autocorrelation accuracy and number of samples $N$, usually requires less operations than the filtering method. Young & Beaulieu (Y&B) proposed a modification to this method which significantly improves its computational efficiency \cite{11}. In \cite{9,11} the IDFT method is proposed and analyzed in the context of Rayleigh multipath (flat) fading channel simulation under Clarke’s two-dimensional (2-D) isotropic scattering model. According to this model, the inphase and quadrature components of the fading process are uncorrelated, each

\textsuperscript{1}Except whenever indicated, it is assumed that the Gaussian process to be generated is zero-mean and circularly-symmetric.
with (normalized) autocorrelation [12]

\[ R(\tau) = J_0(2\pi f_d \tau), \quad -\infty < \tau < \infty \]  
(1a)

and band-limited PSD

\[ S(f) = \frac{1}{\pi f_d \sqrt{1 - \left( \frac{f}{f_d} \right)^2}}, \quad |f| < f_d \]  
(1b)

where \( f_d \) is the maximum Doppler frequency and \( J_0(\cdot) \) is the zero-order Bessel function. Because \( R(\tau) \) in (1a) is real and even, \( S(f) \) in (1b) is also real and even. However, the IDFT method may be used to generate Gaussian random variates with arbitrary complex autocorrelation or, equivalently, arbitrary real PSD [due to the Hermitian symmetry \( R(\tau) = R^*(-\tau) \)]. This generalization enables application of the IDFT method to the simulation of nonisotropic fading scenarios in which the autocorrelation of the scatter signal is complex-valued. We will consider the nonisotropic scattering model proposed in [13] which assumes a von Mises (a.k.a. Tikhonov) distribution for the angle of arrival (AoA) of a scatter multipath component. Considering for simplicity that no specular component is present (Rayleigh fading), the probability density function (p.d.f.) of the AoA, \( \theta \), of the scatter component is \( p(\theta) = \frac{\exp(\kappa_p \cos(\theta - \theta_p))}{I_0(\kappa_p)}, \quad |\theta| \leq \pi \), where \( \theta_p = E[\theta], \kappa_p \geq 0 \) is a parameter which controls the width of the distribution and \( I_0(\cdot) \) is the zero-order modified Bessel function. The (complex) autocorrelation and the (real) PSD of the scatter process are given respectively\(^2\) as [13, eq. (2) and (3)]

\[ R(\tau) = \frac{I_0 \left( \sqrt{\kappa_p^2 - (2\pi f_d \tau)^2} + j4\pi \kappa_p \cos(\theta_p) f_d \tau \right)}{I_0(\kappa_p)}, \quad -\infty < \tau < \infty \]  
(2a)

and

\[ S(f) = \frac{\exp(\kappa_p \cos(\theta_p) \frac{f}{f_d}) \cosh \left( \kappa_p \sin(\theta_p) \sqrt{1 - \left( \frac{f}{f_d} \right)^2} \right)}{\pi f_d \sqrt{1 - \left( \frac{f}{f_d} \right)^2} I_0(\kappa_p)}, \quad |f| < f_d \]  
(2b)

This general model encompasses Clarke’s 2-D isotropic fading model in (1a) and (1b) as a particular case for \( \kappa_p = 0 \). The IDFT generation of a random process with slowly decaying autocorrelation [as is the case with (1a) and (2a) for which the envelope decreases only with \( \sqrt{\tau} \) as \( \tau \to \infty \)] is notoriously difficult and, more importantly and due to time-aliasing, there is a fundamental limit on the autocorrelation accuracy attainable with this method. To partially

\(^2\)If a specular component is also present (Rician fading), the IDFT method may be applied as well, but using instead the corresponding autocorrelation and PSD given as [13, eq. (8) and (9)] respectively.

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overcome these difficulties we propose the use of a new frequency mask, which is determined by matching the actual (theoretical) autocorrelation obtained with the IDFT method to an arbitrary target autocorrelation. The report is organized as follows. The derivation of the new frequency mask is presented in Section 2 together with a brief overview of the IDFT method. Section 3 presents results which assess and demonstrate the autocorrelation accuracy improvement attained by the new FM. Finally, Section 4 presents some concluding remarks.

2 Derivation of the new frequency mask

In this Section we derive the new frequency mask to be used with the IDFT method. To enable this derivation a succinct overview of this method is first presented.

2.1 The IDFT algorithm

Consider the Fourier pair \( R(\tau) \leftrightarrow S(f) \) where \( R(\tau) \) is an arbitrary, possibly complex, target (reference) autocorrelation function defined for \(-\infty < \tau < \infty\) and \( S(f) \) is the corresponding power spectral density (PSD) which is assumed to be band-limited to \(|f| \leq f_d\) [note that \( S(f) \) is always real and positive]. The autocorrelation is assumed to be representative of a stationary circularly-symmetric zero-mean complex Gaussian process with real and imaginary components each with variance \( \sigma^2 \), so \( R(0) = 2\sigma^2 \). The IDFT algorithm attempts to generate a vector \( x = [x_0, x_1, \ldots, x_{N-1}]^T \) of stationary, complex Gaussian random variates with a discrete-time autocorrelation \( \hat{R}_m \) which matches as closely as possible the sampled target autocorrelation \( \{R_m, R(\tau) = mT_s\}_{m=0}^{N-1} \) where \( T_s = 1/f_s \) is the sampling period and \( f_s > 2f_d \). For later reference, we collect these samples in the column vector \( R = [R_0, R_1, \ldots, R_{N-1}]^T \). Let \( Y = [Y_0, Y_1, \ldots, Y_{N-1}]^T \) denote a column vector of stationary, independent, circularly-symmetric, complex Gaussian random variates with zero-mean independent real and imaginary components each with variance \( 1/2 \) (so that \( E[|Y_k|^2] = 1 \)) and \( F = [F_0, F_1, \ldots, F_{N-1}]^T \) denote a column vector of real coefficients which are responsible for introducing correlation among the desired random variates. The IDFT algorithm\(^3\) computes the random Gaussian vector \( x \) with elements

\[
x_n = \frac{1}{N} \sum_{k=0}^{N-1} F_k Y_k e^{j2\pi kn/N}, \quad n = 0, 1, \ldots, N-1
\]

\(3\)Let \( \{z_n\}_{n=0}^{N-1} \equiv z = [z_0, \ldots, z_{N-1}] \) and \( \{Z_k\}_{k=0}^{N-1} \equiv Z = [Z_0, \ldots, Z_{N-1}] \) denote a discrete Fourier transform pair. Throughout, we consider the following definitions and notations: \( Z = \text{DFT}(z) \equiv \{Z_k = \sum_{n=0}^{N-1} z_n e^{-j2\pi kn/N}\}_{k=0}^{N-1} \) is the (direct) discrete Fourier transform (DFT) of \( z \) and \( z = \text{IDFT}(Z) \equiv \{z_n = \frac{1}{N} \sum_{k=0}^{N-1} Z_k e^{j2\pi kn/N}\}_{k=0}^{N-1} \) is the IDFT of \( Z \).
where ⊙ stands for vector element-wise multiplication. A detailed statistical analysis of the process \( \{x_n\} \) has been presented in [11]. Here we state only the following results that are necessary in the sequel:

1. Smith [9] first noted that in order to obtain random variates with an autocorrelation close to \( R(\tau) \), the frequency mask \( F \) must be related to the square-root of the sampled PSD values

\[
S_k \triangleq \alpha S(f)_{f = \frac{k f_s}{N}}, \quad k = -\frac{N}{2} + 1, \ldots, -1, 0, 1, \ldots, \frac{N}{2},
\]

where \( \alpha = N^2 \frac{f_s}{N} = N f_s \) is a power normalizing constant which compensates the \( 1/N \) factor in the IDFT definition and the sampling factor \( N/f_s \). Noting that the \( \{F_k\} \) are defined for \( 0 \leq k \leq N - 1 \), the classical frequency mask is [9, 11]

\[
F = \left[ \sqrt{S_0} \sqrt{S_1} \sqrt{S_2} \cdots \sqrt{S_{\frac{N}{2}}} \sqrt{S_{\frac{N}{2} + 1}} \cdots \sqrt{S_{-2}} \sqrt{S_{-1}} \right]^T
\]

which is real and non-negative.

2. Let \( f_D \triangleq f_d T_s \) denote the frequency \( f_d \) normalized with respect to the sampling frequency \( f_s \). Since \( f_d \) is less than \( \frac{f_s}{2} \), \( S_k \) is non-zero only for \( |k| \leq k_D \) where \( k_D = [f_D N] < \frac{N}{2} \) is the integer closest to \( f_D N \) and the \( \{F_k\} \) are non-zero only for \( k = 0, \ldots, k_D, N - k_D, \ldots, N - 1 \). In [11] the authors always consider \( k_D = [f_D N] \) where \( [x] \) is the floor() function. Instead, we consider that \( f_d \) may only assume a priori one of the discrete values \( \frac{f_s}{N} k \) with \( k = 1, 2, \ldots, \frac{N}{2} - 1 \) so \( f_D N = k \) is always an integer. This eliminates the need for using the floor(), ceiling() or other such function and, in addition and more importantly, ensures that the Doppler frequency used in the generation of the random samples and in the computation of the target autocorrelation are exactly the same.

3. Depending on the particular PSD being considered, some of the FM coefficients may need to be modified, as with the value of \( F_{k_D} \) in [9,11] due to the discontinuity of (1b) [or (2b) for that matter] at \( f = \pm f_d \).

4. If we consider the average of \( \{x_n\} \) involving all possible ensembles of \( \{X_k\} \), then \( E[x_n] = 0 \) because \( E[X_k] = 0 \) by assumption. However, a single realization of \( \{x_n\} \) will have a mean equal to the time average \( E_n[x_n] = \frac{1}{N} \sum_{k=0}^{N-1} F_k X_k E_n \left[ e^{j\frac{2\pi}{N} k n} \right] = \frac{F_0 X_0}{N} \). Therefore, to ensure that every realization of \( \{x_n\} \) has zero-mean, it is necessary to set \( F_0 = 0 \). If a process with non-zero complex mean \( \mu \) is desired (e.g., when generating a Rician fading process), it may be generated either by setting \( F(0) = 0 \) and adding \( \mu \) to each \( x_n \) or, alternatively, as first suggested in [11], by setting \( F_0 X_0 = N \mu \).
5. The autocorrelation of the generated sequence \( x_n = x_n^I + jx_n^Q \) where \( x_n^I \triangleq \Re\{x_n\} \) and \( x_n^Q \triangleq \Im\{x_n\} \) is given as:

\[
\hat{R}_m \triangleq E[x_n x_{n-m}^*] = \frac{1}{N} \sum_{k=0}^{N-1} |F_k|^2 e^{j\frac{2\pi}{N} mk}, \quad m = 0, 1, \ldots, N - 1
\]  

(5)

or, in vector form, as:

\[
\hat{\mathbf{R}} = \frac{1}{N} \text{IDFT}(\mathbf{G})
\]  

(6)

where \( \hat{\mathbf{R}} \triangleq [\hat{R}_0 \ \hat{R}_1 \ \cdots \ \hat{R}_{N-1}]^T \) and \( \mathbf{G} \triangleq \mathbf{F} \odot \mathbf{F} = [G_0 \ G_1 \ \cdots \ G_{N-1}]^T \) with \( \{G_k \triangleq F_k^2 = S_k \geq 0\}_{k=0}^{N-1} \). The correlations among the real and imaginary components of \( x_n \) are given as

\[
E[x_n^I x_{n-m}^I] = E[x_n^Q x_{n-m}^Q] = \frac{\Re\{\hat{R}_m\}}{2}
\]  

(7a)

and

\[
E[x_n^I x_{n-m}^Q] = -E[x_n^Q x_{n-m}^I] = -\frac{\Im\{\hat{R}_m\}}{2}.
\]  

(7b)

In order for each component process \( \{x_n^I\} \) and \( \{x_n^Q\} \) to have variance \( \sigma^2 \) it is required that \( \hat{R}_0 = 2\sigma^2 \) or equivalently, from (5), \( \mathbf{1}^T \mathbf{G} = 2\sigma^2 N \) where \( \mathbf{1} \) is the \( N \times 1 \) column vector \( [1 \ 1 \ \cdots \ 1]^T \). Thus, a given set of coefficients for which this condition does not hold, may require the normalization

\[
\mathbf{G} \leftarrow \mathbf{G} \frac{2\sigma^2 N^2}{\mathbf{1}^T \mathbf{G}}.
\]  

(8)

The following derivation illustrates an important limitation of the IDFT method. Let us consider the samples \( \{S_k\} \) as defined previously in (4) which result from sampling the band-limited PSD \( \alpha S(f) \) at discrete frequencies \( f_k = kf_0 \) with \( f_0 \triangleq \frac{f_s}{N} \) where \( T_0 = NT_s \). The resulting (continuous-frequency) PSD is

\[
S_a(f) = \alpha \sum_{k=-\infty}^{\infty} S(f) \delta(f - kf_0).
\]  

(9)

Using the Poisson sum formula, (9) may be expressed as

\[
S_a(f) = \frac{\alpha}{f_0} \sum_{k=-\infty}^{\infty} S(f) e^{-j2\pi f \frac{k}{f_0}}.
\]  

Taking the inverse Fourier transform \( R_a(\tau) = F^{-1}\{S_a(f)\} \) we obtain

\[
R_a(\tau) = N^2 \sum_{l=-\infty}^{\infty} R(\tau - lNT_s).
\]  

(10)

We now relate (10) to the autocorrelation obtained with the IDFT method. First, rewrite (9) as

\[
S_a(f) = \alpha \sum_{k=-\infty}^{\infty} S(kf_0) \delta(f - kf_0) = \alpha \sum_{k=0}^{N-1} S(kf_0) \delta(f - kf_0)
\]  

where the last equality
follows from the band-limited assumption\footnote{Here, we consider $S(f)$ folded into the interval $0 \leq f \leq f_s$. This has no consequence in the analysis but is required in view of the adopted IDFT definition.} on $S(f)$. Then compute $R_a(\tau) = \mathcal{F}^{-1}\{S_a(f)\} = \int_{-\infty}^{\infty} \alpha \sum_{k=0}^{N-1} S(kf_0) \delta(f - kf_0) e^{2\pi j f \tau} df = \alpha \sum_{k=0}^{N-1} S(kf_0) e^{j2\pi kf_0 \tau}$. Equating this expression for $R_a(\tau)$ to (10) and sampling at instants $\tau = mT_s$, $m = 0, 1, \ldots, N - 1$ we may write

$$\sum_{k=0}^{N-1} \alpha S(kf_0) e^{j2\pi km} = \alpha \sum_{l=-\infty}^{\infty} R(\tau - lNT_s)_{\tau=mT_s}$$

or equivalently

$$\frac{1}{N} \text{IDFT}(|S_k|) = \left\{ \sum_{l=-\infty}^{\infty} R(\tau - lNT_s)_{\tau=mT_s} \right\}_{m=0}^{N-1}$$

which coincides with $\hat{R}_m$ in (6) so

$$\hat{R}_m = R_m + \sum_{l=-\infty, l \neq 0}^{\infty} R(mT_s - lNT_s)_{\hat{\varepsilon}_a(m)}.$$  

Note that as expected $\{\hat{R}_m\}$ is periodic with period $N$. In addition, because $R(\tau)$ is not time-limited, the time-shifted replicas in (13) will overlap and $\hat{R}_m$ will be a (scaled) time-aliased version of $\{R_m = R(\tau)_{\tau=mT_s}\}$. While the overlap is minimized as $N$ increases, an irreducible time-aliasing error $\varepsilon_a(m)$ will always be present in $\hat{R}_m$. For autocorrelations with slow decaying envelope this error may be significant, even for very large $N$. This places a fundamental lower bound on the attainable autocorrelation accuracy, meaning that the desired autocorrelation cannot be synthesized exactly using the IDFT algorithm. The time-aliasing error will be used later in Section 3 as an indication of the matching accuracy between $\hat{R}_m$ and $R_m$ and accordingly, of the merit of a given frequency mask.

### 2.2 New IDFT frequency mask

Instead of using a frequency domain specification based on the sampled PSD, we seek a frequency mask providing a better match between the theoretical IDFT correlation $\{\hat{R}_m\}$ in (5) and the sampled target correlation $\{R_m\}$. The rationale behind this approach is that in some applications it may be more important to have random variates with higher autocorrelation accuracy rather than the higher PSD accuracy provided by the classical approach. This is the case when generating variates for fading channel simulation since important statistics like...
the envelope average fade duration (AFD) or level crossing rate (LCR) depend directly on the autocorrelation and/or its derivatives at low lags \[12\], particularly at \(\tau = 0\). Let \(G_{\text{DFT}}\) denote the set of optimum coefficients for which \(\hat{\mathbf{R}} = \mathbf{R}\) as desired. Then, from (6)

\[
\mathbf{G}_{\text{DFT}} = N \text{DFT}(\mathbf{R}).
\] (14)

Explicitly, the elements of \(\mathbf{G}_{\text{DFT}}\) are computed as

\[
G_{\text{DFT}}(k) = N \sum_{m=0}^{N-1} R_m e^{-j\frac{2\pi}{N} mk}, \quad k = 0, 1, \ldots, N - 1.
\] (15)

Let \(\{\hat{S}_k \triangleq \hat{S}(f)_{f = \frac{k}{f_s}}\}\) denote the sampled PSD corresponding to \(\mathbf{R}\) i.e., \(\{\hat{S}_k = G_{\text{DFT}}(k)\}\). Using the same techniques leading to (13) and the DFT definition, we will derive an expression similar to but dual of (13). Consider the truncated autocorrelation \(R_{T_0}(\tau)\), defined as being equal to the true autocorrelation \(R(\tau)\) in the interval \(|\tau| \leq \frac{T_0}{2} = \frac{NT_s}{2}\) and zero elsewhere. The corresponding Fourier transform is denoted \(S_{T_0}(f)\). If we sample \(R_{T_0}(\tau)\) at instants \(\tau = mT_s, m = 0, 1, \ldots, N - 1\) we obtain

\[
R^0_{T_0}(\tau) = \sum_{m=-\infty}^{\infty} R_{T_0}(\tau) \delta(\tau - mT_s) = \frac{1}{T_s} \sum_{m=-\infty}^{\infty} R_{T_0}(\tau) e^{-j2\pi\frac{mT_s}{T}}.
\] (16)

Taking the Fourier transform \(S^0_{T_0}(f) = \mathcal{F}\{R^0_{T_0}(\tau)\}\) we obtain

\[
S^0_{T_0}(f) = \frac{1}{T_s} \sum_{l=-\infty}^{\infty} S_{T_0}(f - lf_s).
\] (17)

We now relate (17) with the spectrum obtained by the DFT of the target correlation. First rewrite (16) as

\[
R^0_{T_0}(\tau) = \sum_{m=-\infty}^{\infty} R_{T_0}(kT_s) \delta(\tau - mT_s) = \sum_{m=0}^{N-1} R_{T_0}(mT_s) \delta(\tau - mT_s)
\]

where the last equality follows from \(R_{T_0}(\tau)\) being time-limited. Then \(S^0_{T_0}(f) = \int_{-\infty}^{\infty} \sum_{m=0}^{N-1} R_{T_0}(mT_s) \delta(\tau - mT_s) e^{-j2\pi f \tau} d\tau = \sum_{m=0}^{N-1} R_{T_0}(mT_s) e^{-j2\pi f mT_s}\). Equating this expression to (17) and sampling at frequencies \(f = k\frac{f_s}{N}\) we get

\[
\sum_{m=0}^{N-1} R_{T_0}(mT_s) e^{-j\frac{2\pi}{N} km} = \frac{1}{T_s} \sum_{l=-\infty}^{\infty} S_{T_0}(f - lf_s) e^{j\frac{2\pi}{N} l f_s}
\] (18)

or equivalently, noting that \(\{R_{T_0}(mT_s) = R_m\}_{m=0}^{N-1}\)

\[
\text{N DFT}([R_m]) = \sum_{l=-\infty}^{\infty} \alpha S_{T_0}(f - lf_s) = \alpha S_{T_0}\left(k\frac{f_s}{N}\right) + \sum_{l=-\infty, l\neq 0}^{\infty} \alpha S_{T_0}\left(k\frac{f_s}{N} - lf_s\right)
\] (19)

\[\text{As was previously done with } S(f), \text{ here we consider } R_{T_0}(\tau) \text{ folded into the interval } 0 \leq \tau \leq T_0. \text{ This has no consequence in the analysis but is required in view of the adopted DFT definition.}\]
which coincides with (14). As expected, $\{\hat{S}_k\}$ is periodic with period $N$. In addition to the error resulting from the truncation of $R(\tau)$ [which makes $S_{R_0}(f)$ different from the target PSD $S(f)$] and because $S_{R_0}(f)$ is not band-limited, the sampled spectrum $\hat{S}_k$ is also affected by a frequency-aliasing error. Thus, forcing the desired autocorrelation has a detrimental effect on the PSD, as might be expected. This behaviour is dual from that observed when forcing the desired PSD (classical approach) as is seen from (13). In this case however, because $S(f)$ is band-limited to $|f| \leq f_d < \frac{f_s}{2}$ (by assumption), there is no truncation error in $\{\hat{R}_m\}$ but only time-aliasing error. The following remarks address the operationalization of (14) and a required, important modification in the computation of the new coefficients:

1. Due to the Hermitian symmetry of the autocorrelation and to ensure that $G_{\text{DFT}}$ is real, the target autocorrelation vector elements should be specified as

$$R_m = \begin{cases} R(\tau_{mT_s}), & m = 0, 1, \ldots, \frac{N}{2} - 1 \\ \Re \{R \left( \frac{NT_s}{2} \right) \}, & m = \frac{N}{2} \\ R^*_{N-m}, & m = \frac{N}{2} + 1, \ldots, N - 1 \end{cases}.$$  

(20)

In principle, the value $R_{\frac{N}{2}}$ in (20) should have been set equal to the target correlation at lag $m = \frac{N}{2}$ i.e., $R_{\frac{N}{2}} = R \left( \frac{NT_s}{2} \right)$. However, in order for the $\{G_{\text{DFT}}(k)\}$ to be real, $R_{\frac{N}{2}}$ must also be real and therefore we set $\Im \{R_{\frac{N}{2}}\} = 0$ or $R_{\frac{N}{2}} = \Re \{R \left( \frac{NT_s}{2} \right) \}$.

2. Interesting, (14) may be viewed as the solution of the following linear least squares (LS) unconstrained problem. Letting $H = [h_{m,k}]$ denote the $N \times N$ matrix with entries $\{h_{m,k} \triangleq \frac{1}{N} e^{j \frac{2\pi}{N} mk}\}_{m,k=0}^{N-1}$, (5) may be written in matrix form as:

$$\hat{R} = HG.$$  

(21)

As is easy to show, $H^H H = N \left( \frac{1}{N^2} \right)^2 I = \frac{1}{N} I$ where $I$ is the $N \times N$ identity matrix and $(\cdot)^H$ stands for vector or matrix conjugate transpose. Let $G_{\text{DFT}}$ denote the set of optimum coefficients for which $\hat{R} = R$ as desired. The linear LS solution is [14, Sec. 8.4]

$$G_{\text{DFT}} = (H^H H)^{-1} H^H R = N^3 H^H R = N \text{DFT}(R)$$  

(22)

which coincides with (14). This could have been anticipated because this linear LS problem is determined i.e., the number of parameters $\{G_{\text{DFT}}(k)\}$, $N$, equals the number of observations $\{R_m\}$ by construction and $H$ is nonsingular (because it is unitary). In fact we may
set \( \hat{R} = R \) in (21) and invert the equation (which is permitted because \( H \) is nonsingular) to obtain, noting that \( H^{-1} = N^3 H^H \)

\[
G_{\text{DFT}} = H^{-1} R = N^3 H^H R = N \text{DFT}(R).
\] (23)

In this case, the fit between the \( R \) and the linear model \( H G_{\text{DFT}} \) is exact and the LS error is zero.

3. If the target autocorrelation envelope decays sufficiently fast such that \( R(\tau) \approx 0 \) for \( |\tau| > \frac{T_0}{2} \), then \( R_{T_0}(\tau) \approx R(\tau) \) and the truncation effect will be negligible. As a consequence, \( R_{T_0}(\tau) \) is (approximately) a true autocorrelation. As such it is positive-semidefinite and all the \( \{G_{\text{DFT}}(k)\} \) will be nonnegative. If however this condition is not verified, then some of the \( \{G_{\text{DFT}}(k)\} \) may be negative. This is explained by the fact that \( R_{T_0}(\tau) \) is not a true autocorrelation but rather a (sampled) truncated segment of the true autocorrelation \( R(\tau) \). As such, it may not be positive-semidefinite and the corresponding sampled (pseudo) PSD \( \{\hat{S}_k = G_{\text{DFT}}(k)\} \) does not have to be nonnegative. This behaviour poses a major difficulty in the application of the IDFT method because the algorithm requires that \( G_k = F_k^2 \geq 0 \) for all \( k \). In Fig. 1 we plot the normalized \( G_{\text{DFT}} \) resulting from (14) (labeled “Initial FM”) with the target autocorrelation (2a) for two values of the parameter \( \kappa_p \). For \( \kappa_p = 1 \) (large spread of the AoA of scatter component) it is seen from Fig. 1(a) that the coefficients \( G_{\text{DFT}}(k) \) are positive for \( 0 \leq k \leq k_D \) and \( N - k_D \leq k \leq N - 1 \) [corresponding approximately to \( S(f) \) in the frequency interval \( |f| \leq f_d \)]. However, for \( \kappa_p = 10 \) (AoA concentrated around \( \theta = \theta_p \)) the plot in Fig. 1(b) shows that the coefficients in this range assume negative values. In both cases, for \( k_D < k < N - k_D \) [i.e., \( f_d < |f| \leq \frac{f_s}{2} \), the region where \( S(f) \) is zero by assumption] the coefficients oscillate between positive and negative values. We may conclude that some of the coefficients will be negative in frequency intervals where \( S(f) \) is zero or close to zero. Therefore, as a simple yet effective modification to enable the use of the IDFT algorithm, we propose that all coefficients within every frequency interval where the \( G_{\text{DFT}}(k) \) alternate in sign be set equal to the corresponding coefficients from the classical FM. The new set of (nonnegative) coefficients resulting from this modification and normalized according to (8) will be denoted \( G_{\text{new}} \). Fig. 2 shows the resulting \( G_{\text{new}} \) when this procedure is applied to the coefficients in Fig. 1 [normalized according to (8) after setting \( G_{\text{new}}(0) = 0 \)].
Figure 1: Normalized coefficients $G_{\text{DFT}}/(2\sigma^2 N^2)$ and normalized classical FM for the nonisotropic autocorrelation (2a) with $N = 2^{14}$, $k_D = 16$ ($f_D \approx 0.001$) and $\theta_p = 0$, $\kappa_p = 1$: (a) $\kappa_p = 1$ and (b) $\kappa_p = 10$. 
Figure 2: Normalized coefficients $G_{\text{new}}$ and normalized classical FM for the nonisotropic autocorrelation (2a) with $N = 2^{14}$, $k_D = 16$ ($f_D \approx 0.001$) and $\theta_p = 0$, $\kappa_p = 1$ and (b) $\kappa_p = 10$. 
3 Results

In this Section we present some results pertaining to the performance of the new FM and compare it with the performance of the classical FM obtained from frequency samples (or modified samples) of the PSD. Without any loss of generality we set \( \sigma^2 = 1 \). All results were obtained using the Matlab\textsuperscript{®} script presented in Appendix B. Example results are presented for three simulation scenarios involving the generation of a Gaussian process with arbitrary autocorrelation and band-limited PSD.

The first example is the simulation of a Rayleigh fading process under Clarke’s two-dimensional isotropic scattering model introduced in Section 1 with autocorrelation (1a) and PSD (1b). For comparison we consider the Y&B frequency mask \[ G_{Y&B}^k = \begin{cases} 
0, & k = 0 \\
\frac{2\sigma^2N}{\pi f_D} \sqrt{1 - \left( \frac{k}{f_DN} \right)^2}, & k = 1, \ldots, k_D - 1 \\
\frac{2\sigma^2Nk_D}{\pi f_D} \left[ \frac{\pi}{2} - \arctan \left( \frac{k_D - 1}{\sqrt{2k_D - 1}} \right) \right], & k = k_D \\
0, & k = k_D + 1, \ldots, N - k_D - 1 \\
G_{Y&B}^{N-k}, & k = N - k_D, \ldots, N - 1 
\end{cases} \] (24) which, except for \( G_{Y&B}^k \) and \( G_{Y&B}^{N-k} \), coincides with the classical FM. The normalized FM coefficients \( G_{\text{new}}^k = (2\sigma^2N^2) \) resulting from the proposed method are plotted in Fig. 3, together with the corresponding Y&B normalized FM coefficients \( \{ G_{Y&B}^k \} \) for \( N = 2^{14} \) and \( k_D = 82 \) resulting in a normalized Doppler frequency \( f_D = \frac{k_D}{N} \approx 0.005 \). While the Y&B coefficients increase monotonically as \( k \) increases, the new coefficients alternate above and below the Y&B values in a way that could not have been anticipated.

The theoretical and the target autocorrelation (of either the real or imaginary component of the process) using the IDFT method with the new and the Y&B frequency mask are plotted in Fig. 4 again for \( N = 2^{14} \) samples \( k_D = 82 \). As is seen, the autocorrelation using the new FM provides a better match to the desired autocorrelation over the whole lag range \( 0 \leq m < \frac{N}{2} \). This is particularly evident for higher lag values as depicted in the inset in this figure. It is interesting to note that as \( m \) increases, the new FM coefficients give an autocorrelation with an increasing amplitude error but with zero crossings that are very close to the correct values. For large \( m \), the Y&B autocorrelation exhibits a flatter amplitude error but its zero crossings lead the correct values by a significant amount. The result of autocorrelation simulations using an ensemble of \( 10^4 \) realizations are also presented and show a good agreement with the theoretical reference.
Figure 3: The new and the Y&B normalized frequency masks for Clarke’s 2-D isotropic fading model with $k_D = 82$ and $N = 2^{14}$.

Figure 4: Theoretical, target and simulated autocorrelation (real or imaginary) obtained with the IDFT method using the new and the Y&B normalized frequency masks for Clarke’s 2-D isotropic fading model.
To better assess the accuracy provided by the IDFT method, we have computed the average absolute correlation (time-aliasing) error (ACE) defined as

\[
\varepsilon(m) = \frac{1}{2\sigma^2(m+1)} \sum_{i=0}^{m} |\varepsilon_a(i)| = \frac{1}{2\sigma^2(m+1)} \sum_{i=0}^{m} |R_i - \hat{R}_i| (25)
\]

which gives an indication of the autocorrelation matching accuracy up to lag \( m \). For the Clarke’s autocorrelation, the ACE is plotted in Fig. 5 against the sample lag \( m \) for \( N = 2^{16} \) generated samples, for \( f_D = \frac{328}{N} \approx 0.005 \) and also for \( f_D = \frac{655}{N} \approx 0.01 \). As can be seen, the new FM provides an ACE which is always significantly lower than the ACE obtained with the Y&B frequency mask. It is noticeable that as \( f_D \) increases the ACE decreases. This is a consequence of the increased accuracy in the PSD definition resulting from the increase in the value of \( k_D \) (\( N \) is kept fixed).

**Figure 5:** Average absolute aliasing error of the IDFT method using the new and the Y&B normalized frequency masks for Clarke’s 2-D isotropic fading model.

The second example pertains to the generation of correlated additive Gaussian noise for the simulation of digital communication receivers using square-root raised-cosine filters with excess bandwidth \( \beta/(2T) \) Hz where \( T \) is the symbol duration. The raised-cosine (RC) autocorrelation is given as [15, eq. (6.2.24)] with PSD [15, eq. (6.2.23)]. Considering the generation of \( N \) noise samples and an oversampling factor \( L \triangleq T/T_s \) samples per symbol, the classical FM is
\[
G_k^{RC} = \begin{cases} 
2\sigma^2 NL, & k = 0, \ldots, k_D^- \\
2\sigma^2 NL \cos^2 \left[ \frac{\pi}{2\beta} \left( \frac{kL}{N} - \frac{1 - \beta}{2} \right) \right], & k = k_D^- + 1, \ldots, k_D^+ \\
0, & k = k_D^+ + 1, \ldots, N - k_D^+ - 1 \\
G_{N-k}^{RC}, & k = N - k_D^+, \ldots, N - 1 
\end{cases}
\]  
(26)

where \( k_D^- = \left\lfloor (1 - \beta) \frac{N}{2L} \right\rfloor \) and \( k_D^+ = \left\lceil (1 + \beta) \frac{N}{2L} \right\rceil \) where \( \lfloor \cdot \rfloor \) stands for the \text{round()} function.

The ACE is plotted in Fig. 6 for \( N = 2^{14} \) generated samples and for different values of \( \beta \) and oversampling factor \( L \). These results were obtained without setting the first coefficient of either FM to zero. Again, in all cases the new FM performance is superior to the classical FM in (26). For \( \beta = 0 \) the PSD is ideal lowpass, bandlimited to \( |f| = \frac{1}{2T} = \frac{f_s}{2L} \) Hz and the autocorrelation is \( R_m = 2\sigma_m \sin(\pi m / L) \cos(\beta \pi m / L) / (\pi m / L) \) which decays slowly with \( \tau \to \infty \). For \( \beta > 0 \) the autocorrelation decay is faster (with \( -2^2 / \tau^3 \)) as \( \tau \to \infty \) and the corresponding ACE is much smaller, even for small values of \( \beta \) as is visible in the results presented for \( \beta = 0.1 \). For a given \( N \) and \( \beta \) the ACE is smaller for smaller values of \( L \) which lead to higher values of both \( k_D^- \) and \( k_D^+ \) and thus to an increased accuracy in the FM definition. It is worth noting that for smooth-decaying PSDs (i.e., autocorrelation with fast decaying envelope) such as the PSD in this example, the

![Figure 6: Average absolute aliasing error of the IDFT method using the new and the classical normalized frequency masks for the RC autocorrelation.](image)
autocorrelation accuracy obtained with the classical FM becomes much closer to the accuracy with the new FM. Indeed in such cases, the ACE is so small that, from a computational point of view, it is preferable to use the classical FM.

The third example considers the simulation of a fading process according to the nonisotropic scattering model with autocorrelation (2a) and PSD (2b). The classical FM considered in this case is

\[
G_k = \begin{cases} 
0, & k = 0 \\
2\sigma^2 N f_s S\left(\frac{k f_d}{N}\right), & k = 1, \ldots, k_D - 1 \\
2\sigma^2 N^2 \int_{f_d - \frac{k f_d}{N}}^{f_d} S(f)df, & k = k_D \\
0, & k = k_D + 1, \ldots, N - k_D - 1 \\
2\sigma^2 N^2 \int_{-f_d + \frac{k f_d}{N}}^{-f_d} S(f)df, & k = N - k_D \\
2\sigma^2 N f_s S\left(\frac{(k-N) f_d}{N}\right), & k = N - k_D + 1, \ldots, N - 1
\end{cases}
\]  

The values of \(G_{k_D}\) and \(G_{N-k_D}\) are obtained in a similar fashion as in [9, 11]. In the case of \(G_{k_D}\) \((G_{N-k_D})\) the area under \(\alpha S(f)\) in the interval \(f \in [f_d - \frac{k f_d}{N}, f_d]\) \((f \in [-f_d, -f_d + \frac{k f_d}{N}]\)) is equated to the area of a rectangle of width \(\frac{k f_d}{N}\) and height \(G_{k_D}\) \((G_{N-k_D})\). These two values were computed numerically using the modification outlined in Appendix A. The normalized coefficients \(G_{\text{new}}/(2\sigma^2 N^2)\) obtained from (14) and the normalized classical FM \(S_k/(2\sigma^2 N^2)\) are plotted in Fig. 2 for \(\theta_p = 0\) and for two different values of the parameter \(\kappa_p\). The alternating behaviour of the new FM is similar to that observed with the Clarke’s spectrum (c.f. Fig. 3) and its values differ significantly from the classical FM. For \(\kappa_p = 10\) the PSD (2b) becomes sharply concentrated in a narrow region near \(f = f_d\) and practically vanishes outside this region. The theoretical, target and simulated autocorrelation using the IDFT method with the new and the classical FM are plotted in Fig. 7 for \(\kappa_p = 1\). With the new FM, both the autocorrelation of the real component in Fig. 7(a) (although not presented, the results for the autocorrelation of the imaginary component are similar) and the cross-correlation between real and imaginary components in Fig. 7(b) match the target reference better than with the classical FM. With either frequency mask the simulated results agree very well with the theoretical correlations. Results for \(\kappa_p = 10\) are presented in Fig. 8. Although the results are similar to the case \(\kappa_p = 1\), due to the smaller number of non-zero coefficients [c.f. Fig. 2(a)], the matching between the theoretical (or the simulated) and the target autocorrelation is slightly poorer in this case. However, taking into account that this PSD is highly concentrated in a narrow frequency interval, the results may be considered satisfactory.
Figure 7: Theoretical, target and simulated complex autocorrelation obtained with the IDFT method for the nonisotropic fading model with $N = 2^{14}$, $k_D = 16$ ($f_D \approx 0.001$), $\theta_p = 0$ and $\kappa_p = 1$: (a) autocorrelation of the real component and (b) cross-correlation between real and imaginary components.
Figure 8: Theoretical, target and simulated complex autocorrelation obtained with the IDFT method for the nonisotropic fading model with $N = 2^{14}$, $k_D = 16$ ($f_D \approx 0.001$), $\theta_p = 0$ and $\kappa_p = 10$: (a) autocorrelation of the real component and (b) cross-correlation between real and imaginary components.
4 Conclusions

In this report we have presented a method to obtain a new frequency mask for the IDFT algorithm when it is used to generate a vector of stationary, circularly-symmetric, complex Gaussian random variates with arbitrary, possibly complex target autocorrelation. The new frequency mask provides a significant accuracy increase in the synthesized autocorrelation relative to the accuracy obtained with the classical frequency mask, particularly when the target autocorrelation has a slow decaying envelope. In addition, this increased performance is achieved without increasing the online computational complexity of the generating algorithm.
Appendix A

In this appendix we outline the procedure for efficient computation of the numerical integration of the PSD \( S(f) \) in (2b) required to determine \( G_{k_D} \) and \( G_{N-k_D} \) (corresponding to \( f = f_d \) and \( f = -f_d \) respectively) in the case of nonisotropic fading. Consider the Clarke PSD in (1b). The indefinite integral \( I(f) = \int f S(u) \, du \) may be evaluated in closed form as

\[
I(f) = \left[ \frac{\pi}{2} - \arcsin \left( \frac{f}{f_d} \right) \right].
\]

However, if we consider the nonisotropic PSD in (2b) then, except for the case \( \kappa_p = 0 \), for which (2b) becomes (1b), \( I(f) \) can not be expressed in closed form and must be computed numerically. However, because \( S(\pm f_d) = \infty \) a numerical integration algorithm is likely to fail or give incorrect results (with the possible exception of a Gaussian quadrature which behaves remarkably well when integrands exhibit discontinuities). In the following we outline a procedure which circumvents this difficulty. Let us rewrite (2b) as

\[
S(f) = H(f) G'(f)
\]

where

\[
H(f) = \frac{\exp \left( \kappa_p \cos(\theta_p) \frac{f}{f_d} \right) \cosh \left( \kappa_p \sin(\theta_p) \sqrt{1 - \left( \frac{f}{f_d} \right)^2} \right)}{\pi I_0(\kappa_p)} \tag{A.1}
\]

and

\[
G'(f) = \frac{1}{f_d \sqrt{1 - \left( \frac{f}{f_d} \right)^2}} \tag{A.2}
\]

The prime denotes derivative with respect to \( f \) i.e. \( G'(f) \triangleq \frac{dG(f)}{df} \). Using integration by parts we may write

\[
I(f) = \int_f^{f_d} S(u) \, du = \int_f^{f_d} H(u)G'(u) \, du = H(u)G(u) \bigg|_f^{f_d} - \int_f^{f_d} H'(u)G(u) \, du \tag{A.3}
\]

with

\[
H'(f) = \frac{\exp \left( \kappa_p \cos(\theta_p) \frac{f}{f_d} \right) \kappa_p}{\pi I_0(\kappa_p) f_d} \left[ \cos(\theta_p) \cosh \left( \kappa_p \sin(\theta_p) \sqrt{1 - \left( \frac{f}{f_d} \right)^2} \right) \right. \\
- \frac{f}{f_d} \sin(\theta_p) \frac{\sinh \left( \kappa_p \sin(\theta_p) \sqrt{1 - \left( \frac{f}{f_d} \right)^2} \right)}{\sqrt{1 - \left( \frac{f}{f_d} \right)^2}} \bigg] \quad \triangleq U(f) \tag{A.4}
\]
and

\[ G(f) = \arcsin \left( \frac{f}{f_d} \right), \quad (A.5) \]

The integral \( I(f) \) may now be calculated using (A.3), (A.4) and (A.5) because \( G(\pm f_d) \) and \( H(\pm f_d) \) are finite and, noting that \( \lim_{f \to \pm f_d} U(f) = \pm \kappa_p \sin(\theta_p) \), so is \( H'(\pm f_d) \). These particular values are

\[ G(\pm f_d) = \pm \frac{\pi}{2} \quad (A.6) \]

\[ H(\pm f_d) = \frac{\exp(\pm \kappa_p \cos(\theta_p))}{\pi I_0(\kappa_p)} \quad (A.7) \]

and

\[ H'(\pm f_d) = \frac{\exp(\pm \kappa_p \cos(\theta_p)) \kappa_p}{\pi I_0(\kappa_p)} \frac{\kappa_p}{f_d} \left[ \cos(\theta_p \mp \kappa_p \sin(\theta_p)) \right]. \quad (A.8) \]
Appendix B

In this appendix we present a Matlab® script which computes the classical and the new FM and computes a number of statistical characteristics of the generated process by the IDFT method. The script also implements Monte-Carlo simulation.

```matlab
%---------------------------------------------------------------------------
% This script implements the proposed method for improved Gaussian random
% variates generation with the IDFT method
% Author: Goncalo Tavares
% Date. October/2009 (revised April 2010)
%---------------------------------------------------------------------------

clear;
N=2^16; % number of samples
s2=1; % sigma^2
fd=655/N; % normalized Doppler frequency
tp=0; % mean AoA
kp=0; % width parameter of the AoA distribution
beta=0; % excess bandwidth factor (raised-cosine spectrum)
L=16; % interpolation factor (raised-cosine spectrum)
ctype='bessel'; % Bessel autocorrelation (uses Y&B frequency mask)
ctype='raised'; % raised-cosine autocorrelation
ctype='noniso'; % nonisotropic autocorrelation
Sftype='nonz'; % G(1) is not zero
Sftype='zero'; % set G(1) to zero
nd=fd*N;
if ctype=='raised'
    nd1=round(((1-beta)*N/(2*L));
    nd=round((1+beta)*N/(2*L));
    m=0:N-1;
    R=sin(pi*m/L)./(pi*m/L).*cos(pi*beta*m/L)./(1-(2*beta*m/L).^2);
    R(1)=1;
    S=zeros(1,N);
    m=1:nd1+1;
    S(m)=L;
    m=(nd1+2):(nd+1);
    S(m)=L*cos(pi/(2*beta)*((m-1)*L/N-(1-beta)/2)).^2;
    m=N-nd+2:N;
    S(m)=S(N-m+2);
    S=S*N;
end
if ctype=='bessel'
    m=0:N-1;
```

R=besselj(0,2*pi*fd*m);
S=zeros(1,N);
m=0:nd-1;
S(m+1)=1./(pi*fd*sqrt(1-(m/(fd*N)).^2));
S(nd+1)=nd/(pi*fd)*(pi/2-atan((nd-1)/sqrt(2*nd-1)));
S(N-nd+1)=S(nd+1);
m=(N-nd+1):(N-1);
S(m+1)=1./(pi*fd*sqrt(1-((N-m)/(fd*N)).^2));
S=S*N;
end

if ctype == 'noniso'
    m=0:N-1;
    R=besseli(0,sqrt(kp^2-(2*pi*fd*m).^2+i*4*pi*kp*cos(tp)*fd*m)).../
besseli(0,kp);
S=zeros(1,N);
m=0:nd-1;
S(m+1)=exp(kp*cos(tp)*m/(fd*N)).*cosh(kp*sin(tp)*sqrt(1-(m/(fd*N)).^2))..../ (besseli(0,kp)*pi*fd*sqrt(1-(m/(fd*N)).^2));

% computes the coefficients for k=k_D and k=N-k_D using a Matlab
% build-in numerical quadrature
    a=fd-1/N;
    K=N;
    fdm=fd;
    fdi=fd;
    tmp1=exp(kp*cos(tp))/(2*besseli(0,kp))-(exp(kp*cos(tp)*a/fdm)/... (pi*besseli(0,kp))*cosh(kp*sin(tp)*sqrt(1-(a/fdm).^2)))*asin(a/fdm);
    tmp2=quad('freq_',a,fdi,1e-10,[],fdm,kp,tp);
    S(nd+1)=K*(tmp1-tmp2);
    tmp1=exp(-kp*cos(tp))/(2*besseli(0,kp))-(exp(-kp*cos(tp)*a/fdm)/... (pi*besseli(0,kp))*cosh(kp*sin(tp)*sqrt(1-(a/fdm).^2)))*asin(a/fdm);
    tmp2=quad('freq_',-fdi,-a,1e-10,[],fdm,kp,tp);
    S(N-nd+1)=K*(tmp1-tmp2);
    m=(N-nd+1):(N-1);
    S(m+1)=exp(kp*cos(tp)*(m-N)/(fd*N)).*cosh(kp*sin(tp)*sqrt(1-((m-N)/(fd*N)).^2))... ./ (besseli(0,kp)*pi*fd*sqrt(1-((m-N)/(fd*N)).^2));
S=S*N;
end
R=2*s2*R;
S=2*s2*S;
% R = target autocorrelation
% S = classical frequency mask
% Adjust the values of the target autocorrelation to verify R(N-k)=R^*(k)
for i=2:N/2
R(N-i+2)=conj(R(i));
end
R(N/2+1)=real(R(N/2+1));
G=real(N*fft(R)); % the real operation is required to remove the
% % imaginary component which is nonzero due to roundoff
% computes the new frequency mask
masktype='snew';
G=setcoef(G,S,nd,N,masktype);
if Sftype == 'zero'
    G(1)=0;
    S(1)=0;
end
% Normalization to have R(0)=2\sigma^2
G=G/sum(G)*(2*s2*N^2);
S=S/sum(S)*(2*s2*N^2);
Rs=zeros(1,N); % new theoretical correlation
Rf=zeros(1,N); % correlation with classical FM
n=[0:nd N-nd:N-1];
for m=0:N-1
    Rs(m+1)=sum(G(n+1).*exp(sqrt(-1)*2*pi*n*m/N))/N^2;
    Rf(m+1)=sum(S(n+1).*exp(sqrt(-1)*2*pi*n*m/N))/N^2;
end
t=1:N;
figure(1);
plot(t,real(Rs)/2,t,real(Rf)/2,t,real(R)/2),grid;
xlabel('lag \it m');
title('Theoretical and target autocorrelations (real x real)');
legend('New FM', 'Classical FM', 'Target');
figure(2);
plot(t,imag(Rs)/2,t,imag(Rf)/2,t,imag(R)/2),grid;
xlabel('lag \it m');
title('Theoretical and target crosscorrelations (real x imag)');
legend('New FM', 'Classical FM', 'Target');
% computes the average absolute correlation error (ACE)
en(1)=abs(Rs(1)-R(1));
ef(1)=abs(Rf(1)-R(1));
for i=2:N/2;
    en(i)=((i-1)*en(i-1)+abs(Rs(i)-R(i)))/i;
    ef(i)=((i-1)*ef(i-1)+abs(Rf(i)-R(i)))/i;
end
en=en/(2*s2);
ef=ef/(2*s2);
k=1:N/2;
figure(3);
semilogy(k,en,k,ef),grid;
xlabel('lag \textit{m}');
title('Average absolute correlation errors (ACE)');
legend('New FM', 'Classical FM');

%------------------------------------------------------
% simulation
%------------------------------------------------------
Ssim=sqrt(S); % classical mask for simulation
Gsim=sqrt(G); % new mask for simulation
NS=10000; % simulation ensemble size
XL=N/2-1; % correlation lag range
Scorr_i=zeros(1,2*XL+1); % (real x real) classical FM
Scorr_q=zeros(1,2*XL+1); % (imag x imag) classical FM
Scorr_iq=zeros(1,2*XL+1); % (real x imag) classical FM
Gcorr_i=zeros(1,2*XL+1); % (real x real) new FM
Gcorr_q=zeros(1,2*XL+1); % (imag x imag) new FM
Gcorr_iq=zeros(1,2*XL+1); % (real x imag) new FM
s=sqrt(1/2);
for i=1:NS
    gn=s*(randn(1,N)+sqrt(-1)*randn(1,N));
    zf=ifft(gn.*Ssim);
    Scorr_i=Scorr_i+xcorr(real(zf),real(zf),XL,'unbiased')/NS;
    Scorr_q=Scorr_q+xcorr(imag(zf),imag(zf),XL,'unbiased')/NS;
    Scorr_iq=Scorr_iq+xcorr(real(zf),imag(zf),XL,'unbiased')/NS;
    zf=ifft(gn.*Gsim);
    Gcorr_i=Gcorr_i+xcorr(real(zf),real(zf),XL,'unbiased')/NS;
    Gcorr_q=Gcorr_q+xcorr(imag(zf),imag(zf),XL,'unbiased')/NS;
    Gcorr_iq=Gcorr_iq+xcorr(real(zf),imag(zf),XL,'unbiased')/NS;
end
tau=(-XL:XL);
k=N/2:(N-1);
figure(4);
plot(tau(k),Gcorr_i(k),tau(k),Scorr_i(k),tau(k),real(R(k-N/2+1))/2),grid
xlabel('lag \textit{m}');
title('Simulated and target autocorrelations (real x real)');
legend('New FM', 'Classical FM', 'Target');
figure(5);
plot(tau(k),Gcorr_iq(k),tau(k),Scorr_iq(k),tau(k),-imag(R(k-N/2+1))/2),grid
xlabel('lag \textit{m}');
title('Simulated and target cross correlations (real x imag)');
legend('New FM', 'Classical FM', 'Target');
%---------------------------------------------------------------------------
% integrand function S(f) for nonisotropic fading
function S=freq_(f,fd,kp,tp)
tmp1=cos(tp)*cosh(kp*sin(tp)*sqrt(1-(f/fd).^2));
tmp2=sin(tp)*(f/fd).*sinh(kp*sin(tp)*sqrt(1-(f/fd).^2))./sqrt(1-(f/fd).^2);
i=find(f==fd);
tmp2(i)=kp*sin(tp);
i=find(f==-fd);
tmp2(i)=-kp*sin(tp);
S=kp*exp(kp*cos(tp)*f/fd)/(pi*fd*besseli(0,kp)).*(tmp1-tmp2).*asin(f/fd);

% Computes the new frequency mask
% masktype == 'zero': the coefficients of the new FM where G oscilates about zero are set to zero
% masktype == 'snew': the coefficients of the new FM where G oscilates about zero are set equal to the corresponding coefficients from the classical FM
function y=setcoef(G,S,nd,N,masktype);
tmp=sign(G);
tmp(find(tmp==0))=-1;
st=0;
j=1;
tmp(N+1)=-tmp(N);
for i=1:N;
    tmp(i)=tmp(i)+tmp(i+1);
    if tmp(i)>0 && st==0
        st=1;
    end
    if tmp(i)==0 && st==1
        tmp(i)=1;
        st=0;
    end
end
i=find(tmp>0);
i(i(find(i<=N)))=;
if type =='zero'
    y=zeros(1,N);
elseif type =='snew'
    y=S;
end
y(i)=G(i);
y((nd+3):(N-nd-3))=0;
References


