Two-Input Fuzzy TPE Systems♦

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Abstract. Single input Fuzzy TPE systems, as proposed by Sudkamp and Hammel, allow a more efficient computational inference of a single input fuzzy rule base. This paper shows that it is possible to generalize single input Fuzzy TPE systems to 2-input systems, therefore extending its range of possible applications. The paper presents the details and proof of the extension validity, and shows benchmark results comparing the 2-input Fuzzy TPE with classical inference systems.

Keywords: 2-input fuzzy TPE, fuzzy inference computational efficiency.

1 Introduction

Sudkamp and Hammell's Fuzzy TPE systems [1] provide a way to accelerate the fuzzy inference procedure in 1-input fuzzy systems. It also makes the inference process independent from the number of involved linguistic terms, since instead of a sequential application of all rules in a rule base, it allows the direct access to the relevant rules using an indexation process and infers the rule result using previously computed constants. This approach is possible due to restrictions imposed on the involved linguistic terms: all membership functions must be triangular, complementary, and evenly spaced in the universe of discourse (UoD). However, Sudkamp's approach was developed for single-input systems, and could not be directly applied in systems with more inputs. Therefore its use was limited to a few particular problems. Through the years TPE systems have been used [2][3], but when more than one input is deemed necessary, alternative inference methods that do not implement a proper fuzzy inference mechanism, like FAMs, are used instead. By proper fuzzy inference mechanisms one means mechanisms that although computationally faster, do not produce the exact same result of classical fuzzy rule base inference processes. This paper shows that it is possible to generalize single input Fuzzy TPE systems to 2-input systems extending range of possible Fuzzy TPE systems applications.

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2 Fuzzy TPE Systems

Sudkamp and Hammell’s model is based on an even triangular partition of a single variable fuzzy universe of discourse (UoD), hence the name TPE – Triangular Partition Evenly. The model bases its efficiency on the symmetry of the straight lines used to define the triangular membership functions. This section summarizes Sudkamp and Hammel’s fuzzy TPE systems.

In this model, the membership function (mbf) of a linguistic term A_i can be defined by (1), where $\mu_{A_i}(x)$ is the membership of crisp input x in A_i, and $\mu_{A_i}(a_i) = 1$.

$$\mu_{A_i}(x) = \begin{cases} (x - a_{i-1}) / (a_i - a_{i-1}) & if\ a_{i-1} \le x \le a_i \\ (-x + a_{i+1}) / (a_{i+1} + a_i) & f\ a_i \le x \le a_{i+1} \\ 0 & otherwise \end{cases} . \quad (1)$$

On a n-linguistic term fuzzy variable, all mbf are equal with the exception of the lower and upper linguistic terms. The mbf of these linguistic terms are respectively truncated to the left and to the right of their a_i points, i.e., $\mu_{A_1}(-1) = 1$ and $\mu_{A_n}(1) = 1$.

An additional condition is that all mbf are complementar in all UoD, i.e.:

$$\sum_{i=1}^{n} \mu_{A_i}(x) = 1 \quad , x \in UoD . \quad (2)$$

Equation (2) ensures the completeness of the rule base. Fig.1 shows a seven fuzzy linguistic term TPE set.

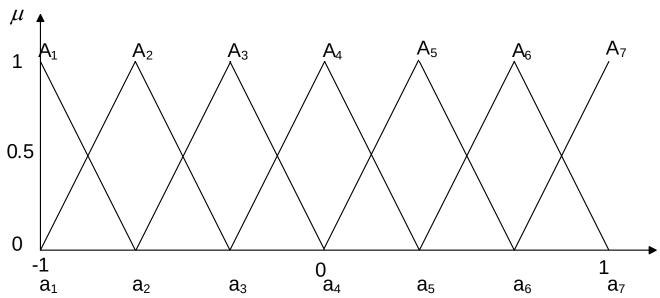

Fig. 1 – Example of mbf that comply with Fuzzy TPE systems’ restrictions

Within a fuzzy TPE system, one can divide the UoD into a set of $[a_i, a_{i+1}]$ intervals where $\mu_{A_i}(x) = 1 - \mu_{A_{i+1}}(x)$, $\mu_{A_i}(a_i) = 1$ and $\mu_{A_j}(a_i) = 0$ for $j \ne i$. Therefore, the membership degree within the interval can be defined by the two following straight line equations:

$$\mu_{A_i}(x) = \frac{-x + a_{i+1}}{a_{i+1} - a_i}. \tag{3}$$

$$\mu_{A_{i+1}}(x) = \frac{x - a_i}{a_{i+1} - a_i}. \tag{4}$$

Within a single-input/single-output TPE rule based fuzzy system, the fuzzy rules that involve antecedents A_i and A_{i+1} can be expressed as:

$$if\ X\ is\ A_i\ then\ Z\ is\ C_r$$
$$if\ X\ is\ A_{i+1}\ then\ Z\ is\ C_s\ ,$$

where C_r and C_s are also TPE linguistic terms. Defining c_r and c_s as the central points of C_r and C_s, and by using weighted averaging defuzzification, the result *z* specified by crisp input *x* is given by:

$$z = \frac{\mu_{A_i}(x)c_r + \mu_{A_{i+1}}(x)c_s}{\mu_{A_i}(x) + \mu_{A_{i+1}}(x)}. \tag{5}$$

By replacing (3) and (4) in (5) and simplifying one obtains:

$$z = \frac{x(c_s - c_r) + a_{i+1}c_r - a_i c_s}{a_{i+1} - a_i} = x\frac{c_s - c_r}{a_{i+1} - a_i} + \frac{a_{i+1}c_r - a_i c_s}{a_{i+1} - a_i}. \tag{6}$$

Equation (6) shows that only two constants are needed to compute *z* given any crisp input $x \in [a_i, a_{i+1}]$. These constants, $(c_s - c_r)/(a_{i+1} - a_i)$ and $(a_{i+1}c_s - a_i c_r)/(a_{i+1} - a_i)$, are determined by the fuzzy rule base and the midpoints of the triangular membership functions, and therefore independent of *x*. Thus a fuzzy TPE system can be represented by a table containing the constants associated with each input interval. Since all intervals are equal in size, one can directly address the constants associated with a given input *x* by using the function

$$trunc(\frac{2x + 2}{n - 1}) + 1. \tag{7}$$

On a fuzzy TPE system one can skip the inference of all rules in the rule base. Given input *x*, all that is needed is to get the appropriate constants from a table and apply (6) to find defuzzified output *z*. Therefore, inference time on a TPE fuzzy rule base is independent of the number of rules and linguistic term size, and TPE fuzzy inference is particularly well adapted to deal with large fuzzy rule bases.

3 Two-input Fuzzy TPE Systems

A single input variable fuzzy rule based system has an obviously limited interest. Multiple input Fuzzy TPE systems are necessary if one wants to use these mechanisms in a significant range of applications. Although it is theoretically possible to extend the model to a higher number of inputs, in this paper we limit the extension to 2-input systems. One must note that in rule based fuzzy systems it is sometimes possible to organize a n-input rule base into several connected 2-input rule bases. This manipulation is particular to the system being modeled and relies on the interdependence of its input variables. Therefore, let us focus on the 2-input case.

Any 2-input fuzzy system rule base with n-linguistic term variables can be represented as a 2 dimensional table. Table 1, represents an excerpt of a generic 2-input fuzzy rule base, where all A_i, B_j and C_x are fuzzy linguistic terms defined by a fuzzy membership function. On such a system, any combination of input values activates at most four different rules of the rulebase.

Table 1. An excerpt of a 2-input Fuzzy Rule Base.

Y \ X	A_i	A_{i+1}
B_j	C_r	C_t
B_{j+1}	C_s	C_u

$$\textit{if } X \textit{ is } A_i \textit{ and } Y \textit{ is } B_j \textit{ then } Z \textit{ is } C_r$$

$$\textit{if } X \textit{ is } A_i \textit{ and } Y \textit{ is } B_{j+1} \textit{ then } Z \textit{ is } C_s$$

$$\textit{if } X \textit{ is } A_{i+1} \textit{ and } Y \textit{ is } B_j \textit{ then } Z \textit{ is } C_t$$

$$\textit{if } X \textit{ is } A_{i+1} \textit{ and } Y \textit{ is } B_{j+1} \textit{ then } Z \textit{ is } C_u$$

Since TPE’s membership functions are, by definition, triangular and symmetric, Table 1 can be replaced by a Numeric Inference (NI) table, which is an alternative representation where each linguistic term is replaced by its central point *x*-coordinate. Therefore, if one assumes the weighted averaging defuzzification method, one can deduce the following output equation for the rule base presented in Table 1, where c_r, c_s, c_t and c_u are the central point x-coordinates of the consequent mbfs:

$$z = \frac{\min(\mu_{A_i}(x), \mu_{B_j}(y))c_r + \min(\mu_{A_i}(x), \mu_{B_{j+1}}(y))c_s + \min(\mu_{A_{i+1}}(x), \mu_{B_j}(y))c_t + \min(\mu_{A_{i+1}}(x), \mu_{B_{j+1}}(y))c_u}{\min(\mu_{A_i}(x), \mu_{B_j}(y)) + \min(\mu_{A_i}(x), \mu_{B_{j+1}}(y)) + \min(\mu_{A_{i+1}}(x), \mu_{B_j}(y)) + \min(\mu_{A_{i+1}}(x), \mu_{B_{j+1}}(y))} \quad (8)$$

In order to find the 2-dimensional equivalent of (6), consider two fuzzy variables *A* and *B* defined respectively by *n* and *m* TPE mbf. Fig.2 shows a pictorial representation of *A* and *B*, where *x* and *y* are the crisp input values.

The 2-dimensional intervals we consider to find the 2-dimensional equation equivalent of (6) are represented by $\left[a_i, a_{i+1}\right]$ and $\left[b_j, b_{j+1}\right]$. The shadowed region in Fig.2 shows a 2-dimensional interval. 2-dimensional interval indexation is based on (7), and results on (9):

$$interval_A = \begin{cases} n-1 & ,if \;\; x=1 \\ trunc\left(\dfrac{(x+1)(n-1)}{2}\right)+1 & ,if \;\; x \neq 1 \end{cases} \tag{9}$$

$$interval_B = \begin{cases} m-1 & ,if \;\; y=1 \\ trunc\left(\dfrac{(y+1)(m-1)}{2}\right)+1 & ,if \;\; y \neq 1 \end{cases}$$

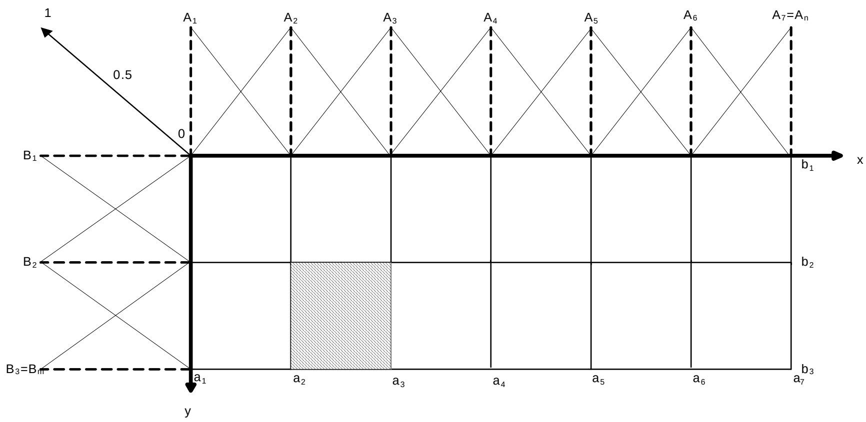

Fig.2 – Two-dimensional representation of TPE linguistic terms. The shadowed region represents a 2-dimensional interval

One can divide each 2-dimensional interval in four different areas, as depicted in Fig.3. Each of these areas is characterized by a constant relation between the membership degrees of *x* and *y* in the pertaining linguistic terms. For example, in Area 1, all the following equations hold true:

$$\min(\mu_{A_i}(x), \mu_{B_j}(y)) = \mu_{B_j}(x),$$

$$\min(\mu_{A_i}(x), \mu_{B_{j+1}}(y)) = \mu_{B_{j+1}}(x),$$

$$\min(\mu_{A_{i+1}}(x), \mu_{B_j}(y)) = \mu_{A_{i+1}}(x),$$

$$\min(\mu_{A_{i+1}}(x), \mu_{B_{j+1}}(y)) = \mu_{A_{i+1}}(x).$$

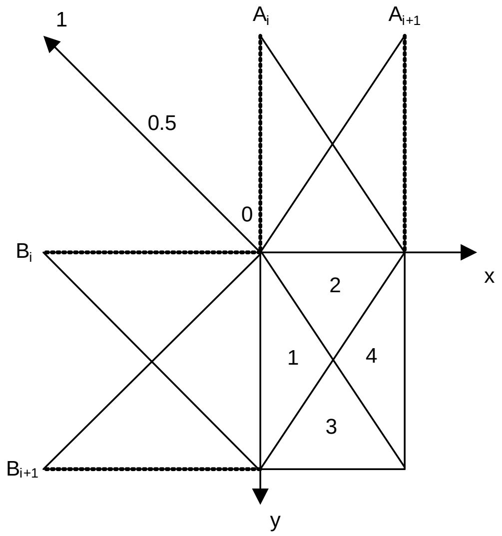

Fig. 3 – The 4 different areas in each interval

Based on the characteristics of each of the 4 areas, it is possible to divide the defuzzification equation (8) into 4 different defuzzification equations, where the min() function is not necessary.

The 4 areas can be divided by two straight lines defined by the following equations:

$$y = \frac{(n-1)(x+1)}{m-1} + 2\times\left(\frac{interval_B - 1}{m-1} - \frac{interval_A - 1}{n-1}\right) - 1. \tag{10}$$

$$y = \frac{2-(n-1)(x+1)}{(m-1)} + 2\times\left(\frac{interval_B - 1}{m-1} + \frac{interval_A - 1}{n-1}\right) - 1. \tag{11}$$

Equations (10) and (11) can be used to create Table2, which divides the Universe of Discourse in intervals, and each interval in the 4 different areas.

Table 2.– Area definition for each interval

	$y < \frac{(n-1)(x+1)}{(m-1)} + 2\times\left(\frac{interval_B - 1}{m-1} - \frac{interval_A - 1}{n-1}\right) - 1$	$y \geq \frac{(n-1)(x+1)}{(m-1)} + 2\times\left(\frac{interval_B - 1}{m-1} - \frac{interval_A - 1}{n-1}\right) - 1$
$y \geq \frac{2-(n-1)(x+1)}{(m-1)} + 2\times\left(\frac{interval_B - 1}{m-1} + \frac{interval_A - 1}{n-1}\right) - 1$	Area 1	Area 3
$y < \frac{2-(n-1)(x+1)}{(m-1)} + 2\times\left(\frac{interval_B - 1}{m-1} + \frac{interval_A - 1}{n-1}\right) - 1$	Area 2	Area 4

Since we are dealing with a TPE system, (3) and (4) are still valid for each fuzzy variable. Therefore, $\mu_{A_i}(x)$, $\mu_{A_{i+1}}(x)$, $\mu_{B_j}(y)$, $\mu_{B_{j+1}}(y)$ can be defined by:

$$\begin{aligned} \mu_{A_i}(x) &= \frac{-x+a_{i+1}}{a_{i+1}-a_i} \quad & \mu_{B_j}(y) &= \frac{-y+b_{j+1}}{b_{j+1}-b_j} \\ \mu_{A_{i+1}}(x) &= \frac{x-a_i}{a_{i+1}-a_i} \quad & \mu_{B_{j+1}}(y) &= \frac{y-b_j}{b_{j+1}-b_j} \end{aligned} . \tag{12}$$

Considering the 4 different areas defined in Fig.3 for each interval, one can replace (12) in (8) and manipulate the resulting expressions in order obtain the following defuzzification equations:

Area 1: (13)

$$z_1 = \frac{\frac{-y+b_{j+1}}{b_{j+1}-b_j}c_r + \frac{y-b_j}{b_{j+1}-b_j}c_s + \frac{x-a_i}{a_{i+1}-a_i}c_t + \frac{x-a_i}{a_{i+1}-a_i}c_u}{\frac{-y+b_{j+1}}{b_{j+1}-b_j} + \frac{y-b_j}{b_{j+1}-b_j} + \frac{x-a_i}{a_{i+1}-a_i} + \frac{x-a_i}{a_{i+1}-a_i}} = \frac{y\frac{(c_s-c_r)}{b_{j+1}-b_j} + \frac{b_{j+1}c_r - b_j c_s}{b_{j+1}-b_j} + x\frac{(c_t+c_u)}{a_{i+1}-a_i} + \frac{-a_i(c_t+c_u)}{a_{i+1}-a_i}}{y.0 \;+\; 1 \;+ x\frac{2}{a_{i+1}-a_i} + \frac{-2a_i}{a_{i+1}-a_i}}$$

Area 2: (14)

$$z_2 = \frac{\frac{-x+a_{i+1}}{a_{i+1}-a_i}c_r + \frac{y-b_j}{b_{j+1}-b_j}c_s + \frac{x-a_i}{a_{i+1}-a_i}c_t + \frac{y-b_j}{b_{j+1}-b_j}c_u}{\frac{-x+a_{i+1}}{a_{i+1}-a_i} + \frac{y-b_j}{b_{j+1}-b_j} + \frac{x-a_i}{a_{i+1}-a_i} + \frac{y-b_j}{b_{j+1}-b_j}} = \frac{y\frac{(c_s+c_u)}{b_{j+1}-b_j} + \frac{-b_j(c_s+c_u)}{b_{j+1}-b_j} + x\frac{(c_t-c_r)}{a_{i+1}-a_i} + \frac{a_{i+1}c_r - a_i c_t}{a_{i+1}-a_i}}{y\frac{2}{b_{j+1}-b_j} + \frac{-2b_j}{b_{j+1}-b_j} + \;x.0\; + \;1}$$

Area 3: (15)

$$z_3 = \frac{\frac{-y+b_{j+1}}{b_{j+1}-b_j}c_r + \frac{-x+a_{i+1}}{a_{i+1}-a_i}c_s + \frac{-y+b_{j+1}}{b_{j+1}-b_j}c_t + \frac{x-a_i}{a_{i+1}-a_i}c_u}{\frac{-y+b_{j+1}}{b_{j+1}-b_j} + \frac{-x+a_{i+1}}{a_{i+1}-a_i} + \frac{-y+b_{j+1}}{b_{j+1}-b_j} + \frac{x-a_i}{a_{i+1}-a_i}} = \frac{y\frac{-(c_r+c_t)}{b_{j+1}-b_j} + \frac{b_{j+1}(c_r+c_t)}{b_{j+1}-b_j} + x\frac{(c_u-c_s)}{a_{i+1}-a_i} + \frac{a_{i+1}c_s - a_i c_u}{a_{i+1}-a_i}}{y\frac{-2}{b_{j+1}-b_j} + \frac{2b_{j+1}}{b_{j+1}-b_j} + \;x.0\; + \;1}$$

Area 4: (16)

$$z_4 = \frac{\frac{-x+a_{i+1}}{a_{i+1}-a_i}c_r + \frac{-x+a_{i+1}}{a_{i+1}-a_i}c_s + \frac{-y+b_{j+1}}{b_{j+1}-b_j}c_t + \frac{y-b_j}{b_{j+1}-b_j}c_u}{\frac{-x+a_{i+1}}{a_{i+1}-a_i} + \frac{-x+a_{i+1}}{a_{i+1}-a_i} + \frac{-y+b_{j+1}}{b_{j+1}-b_j} + \frac{y-b_j}{b_{j+1}-b_j}} = \frac{y\frac{(c_u-c_t)}{b_{j+1}-b_j} + \frac{b_{j+1}c_t - b_j c_u}{b_{j+1}-b_j} + x\frac{-(c_r+c_s)}{a_{i+1}-a_i} + \frac{a_{i+1}(c_r+c_s)}{a_{i+1}-a_i}}{y.0 \;+\; 1 \;+ x\frac{-2}{a_{i+1}-a_i} + \frac{2a_{i+1}}{a_{i+1}-a_i}}$$

Note that all 4 equations can be expressed under the form

$$z = \frac{yH_1 + H_2 + xH_3 + H_4}{yH_5 + H_6 + xH_7 + H_8} , \tag{17}$$

where all H_i are constants.

As in the single input case, the H_i constants are determined by the fuzzy rule base and by the central point of each linguistic term (a_i, b_j, c_l), and are independent of input crisp values x and y. Therefore, they can be computed only once, prior to the actual rule base inference process.

Based on (9) and Table 2, one can directly index an inference matrix with dimensions $[n\text{-}1]\times[m\text{-}1]\times[4]\times[8]$, where in each position of the inference matrix one will have the previously defined H constant. From this inference matrix, one can directly compute the defuzzified output z. By using this inference process, one avoids the sequential computation of each single rule while simultaneously simplifying the defuzzification process. Therefore, as long as all H_i are previously computed, and with proper indexation of the input values, it is possible to obtain a very fast fuzzy inference process that does not grows exponentially with the increase of number of linguistic terms in each involved fuzzy variable.

4 Results

In order to test the computational efficiency of 2-Input Fuzzy TPE Systems, a comparison was made with traditional fuzzy rule based inference methods.

Since the Fuzzy TPE method computational inference time does not grow exponentially with the increase of number of linguistic terms in each involved fuzzy variable (contrary to the traditional methods), better results were to be expected when a larger number of membership functions were used. Therefore tests were made involving fuzzy variables with 3 and 7 linguistic terms, resulting in complete rule bases with 9 and 49 rules respectively. Table 3. and Table 4. show the used rule bases. Note that inference computational time is independent of actual rule content.

Table 3.– Fuzzy rule base for 3 linguistic term's input variables

Y \ *X*	*L*	*M*	*H*
L	*L*	*M*	*H*
M	*M*	*H*	*M*
H	*H*	*M*	*L*

For the traditional fuzzy inference method computational implementation, one opted to use an individual array to represent each linguistic term membership function. This method is computationally faster than using straight line equations, since inferring $\mu_{A_i}(x)$ is done via a simple array indexation.

Table 4.– Fuzzy rule base for 7 linguistic term's input variables

Y \ X	*VVL*	*VL*	*L*	*M*	*H*	*VH*	*VVH*
VVL	*VVL*	*VL*	*L*	*M*	*H*	*VH*	*VVH*
VL	*VL*	*L*	*M*	*H*	*VH*	*VVH*	*VH*
L	*L*	*M*	*H*	*VH*	*VVH*	*VH*	*H*
M	*M*	*H*	*VH*	*VVH*	*VH*	*H*	*M*
H	*H*	*VH*	*VVH*	*VH*	*H*	*M*	*L*
VH	*VH*	*VVH*	*VH*	*H*	*M*	*L*	*VL*
VVH	*VVH*	*VH*	*H*	*M*	*L*	*VL*	*VVL*

Using this method, the C code to compute a fuzzy rule like "If *x* is *A* and *y* is *B* Then *Z* is *C*", can be resumed as:

```
miu=MIN(is(A,x),is(B,y));  //If x is A And y is B
if (miu!=0){
    for (n=0; n<number_of_mbf; n++)   //Then z is C
         foutput[n] += miu*C[n];
}
```

where function 'is()' is simply:

```
float is(int LT, int inp){ //LT-Linguistic Term,inp-input
     return mbf[LT][inp];} //mbf-Array of all LT
```

The above mentioned code is repeated for each rule in the rule base. As a result one obtains an array representing fuzzy variable *z*.

The defuzzification method used is weighted averaging, which is also simple to implement:

```
for  (n=0;n<mbfsize;n++){
      mass+=foutput[n]*n;
      area+=foutput[n];
}
z=mass/area;
```

Although the defuzzification code is simple, it is not computationally efficient since the number of rules grows exponentially with the increase in the number of linguistic terms, and the code uses several cycles to run through the rule base. It is possible to vastly improve defuzzification computation time, but not without imposing restrictions to mbf shape (which is what fuzzy TPE does).

The Fuzzy TPE method accesses the active rules through direct indexation, and therefore avoids the necessity to test each single rule. Due to the restrictions in the shape of the membership functions, it also uses a much more efficient defuzzification method.

Table 5. shows the average inference computing time for the rule bases in Table 3. and Table 4.

Table 5.– Average rule base inference computing time comparison (ns)

Method / #LT/#rules	Classical Inference	Fuzzy TPE
3 / 9	44.4	6.6
7 / 49	58.7	7.2

The average inference time was obtained after computing 500000 inputs on a 1.86GHz PentiumM processor. One can see that Fuzzy-TPE inference is roughly 8 times faster than a classical fuzzy inference method, which can be considered a considerable improvement. However, one must not ignore that the involved computing times are so small that can often be considered a negligible factor in many applications. Therefore, due to the restrictions imposed to the linguistic terms, the use of 2-input Fuzzy-TPE systems can only be justified in applications where a large volume of information must be processed and where time is a major issue, like real-time control systems or computer chess.

5 Conclusions and Future Developments

It was shown that two-input Fuzzy TPE inference systems can be implemented and that they provide a significant improvement over classical fuzzy inference systems in what concerns computing performance. However, this improvement can only be useful in systems where time and performance are critical and where the restrictions imposed to the linguistic term's membership functions can be accepted.

N-input Fuzzy TPE systems are theoretically possible. However, as the number of inputs increases, the complexity of the defuzzification equation and the dimensionality of the computing matrixes increase exponentially. Therefore, n-input Fuzzy TPE systems viability over classical inference systems still needs to be proved on a future work.

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