Pseudo-Fractional ARMA modelling using a double Levinson recursion

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Abstract – In this paper the modeling of Fractional Linear Systems through ARMA models is addressed. To perform this study a new recursive algorithm for Impulse Response ARMA modelling is presented. This is a general algorithm that allows the recursive construction of ARMA models from the Impulse Response sequence. This algorithm does not need an exact order specification, since it gives some insights into the correct orders. It is applied to modelling Fractional Linear Systems described by fractional powers of the backward difference and the bilinear transformations. The analysis of the results leads us to propose suitable models for those systems.

I. INTRODUCTION

Pseudo-Fractional ARMA modelling is a pole-zero modelling of fractional linear systems. These are described by fractional differential equations in the continuous-time case or ARIMA models in the discrete-time case. The first case is based on the definition of fractional differintegration while the second deals with the fractional differencing that is a fractional version of the well known finite differences. These systems are characterised by having a long memory that cannot be explained by the usual linear systems that are short memory (exponential). The desire of finding a theoretical base for such systems led to the fractional calculus that has recently received a great deal of attention in scientific literature, through the publication of books, special issues of journals, review articles, as well as a very large number of research papers. The interest in the fractional calculus comes from the fact that it provides foundations for the understanding of several natural phenomena and the basic theory for building models for the systems underlying them. However, the adoption of the fractional calculus by the physicians and engineering community was inhibited historically by the lack of clear experimental evidence for its need and by the difficulty in constructing simple models for simulation or even implementation of simple fractional systems. Fractional Calculus is almost as old as the common Calculus, but only since 30 years ago it has been subject of specialized publications and conferences.

The basic building block of this kind of systems is the non integer order derivative and integral that have been approximated by fractional powers of the backward difference or the bilinear transformations – the former is exactly the building block of the fractional differencing, as said above. However, these approximations are described IIR systems with non rational transfer functions. For these, ARMA models are only approximations. However, the usefulness of ARMA models makes them very interesting when constructing discrete-time approximating models for fractional systems. In the last few years a lot of attempts to obtain such models have been done {see [1-5]}. However, it is not clear how to perform

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such modelling neither how to choose the most suitable orders, although there are a lot of algorithms, mainly in the stochastic case. In Impulse Response modelling the well known Padé algorithm is frequently used [6]. In this paper we shall not be concerned with the estimation task involved in ARMA modelling; instead, we are going to look at the underlying structure of the ARMA model in order to find alternative relations to obtain the ARMA parameters from the Impulse Response. To be more specific, consider the usual theoretical approach for computing the ARMA(N,M) parameters from the Impulse Response, h_n:

$$\sum_{i=0}^{N} a_i h_{j-i} = \begin{cases} b_j & j \geq 0; \ldots; M \\ 0 & j>M \end{cases} \tag{1}$$

where a_i and b_i are, respectively, the AR and MA parameters. A close look into equation (1) shows that, to compute the ARMA(N,M) parameters, we only need the first N+M+1 values of the Impulse Response{h(n), n = 0, ... , N+M}. Here we propose a new description of the double Levinson recursion presented in [7]. The algorithm consists of the recursive solution of the system obtained from (1) with j = M to j = N+M for the AR coefficients followed by the use of the first M+1 equations to obtain the MA parameters. This algorithm gives us the possibility of determining the orders of the systems when looking at the pattern formed by a sequence of coefficients. We applied this algorithm in (pseudo) fractional ARMA modelling and we observed that the pattern does not point clearly the orders, but give us some insights into minimum orders.

The recursive algorithm is presented in section II, where we also present two examples of its application. In section III we describe the results concerning fractional modelling. Last but not least, we will present some conclusions in section IV.

II. THE ALGORITHM

The procedure we are going to describe is very similar to the one presented in [7] but, instead of using the matrix formulation, we adopt a Schur-like description [8] since it is more direct and easier to implement. To begin with, we consider (1) and introduce a function $f^{N;M}(j)$, j = 0, 1, … , given by:

$$f^{N}_{M}(j) = \sum_{i=0}^{N} a^{N}_{M}(i)\, h_{j+M-i} \tag{2}$$

where we enhance the orders N and M. According to (1) this function has gaps for j = 1, 2, … N. For N = 0, we thus have:

$$f^{0;M}(j) = h_{j+M} \qquad j = 0, 1, \ldots \tag{3}$$

The algorithm described in [7] uses an adjoint system [9]. Here, we introduce an adjoint function defined by:

$$g^{N}_{M}(j) = \sum_{i=0}^{N} \gamma^{N}_{M}(i)\, h_{N-j+M-i} \tag{4}$$

with

$$g^{0;M}(j) = h_{-j+M} \qquad j = 0, 1, \ldots \tag{5}$$

As it is clear, $g^{N;M}(j)$ has gaps for j = 1, 2,…, N, too. The solution of (1) is recursively constructed for successive values of N from N = 1 to N = N_0, where N_0 is a positive integer. To do this, assume that we have constructed the (N-1)th order functions $f^{N-1;M}(j)$ and $g^{N-1;M}(j)$ j = 0, 1, … We will construct the Nth order functions by the recursions:

$$f^{N;M}(j) = f^{N-1;M}(j) + K^{N;M} g^{N-1;M}(N-j) \tag{6}$$

and

$$g^{N;M}(j) = g^{N-1;M}(j) + H^{N;M} f^{N-1;M}(N-j) \tag{7}$$

where $K^{N;M}$ and $H^{N;M}$ are obtained by forcing both functions to have a gap at j = N. We obtain

$$K_M^N = - \frac{f_M^{N-1}(N)}{g_M^{N-1}(0)} \qquad (8)$$

$$H^{N;M} = - \frac{g_M^{N-1}(N)}{f_M^{N-1}(0)} \qquad (9)$$

As it is easy to verify, we have also:

$$g^{N;M}(0) = f^{N-1;M}(0)\left(1 - K^{N;M}.H^{N;M}\right) \qquad (10)$$

If the system with Impulse Response h_n is really an ARMA(N_0,M_0), we will have

$$b_j = f^{N_0;M_0}(j - M_0) \qquad j=0, \dots, M_0 \qquad (11)$$

For the AR coefficients we use (2), (4), (6), and (7) and the $K^{N;M}$ and $H^{N;M}$ sequences to obtain the so-called Double Levinson recursion [7]:

$$a^{N;M}(i) = a^{N-1;M}(i) + K^{N;M}\,\gamma^{N-1;M}(N-i) \qquad (12)$$

and

$$\gamma^{N;M}(i) = \gamma^{N-1;M}(i) + H^{N;M}\,a^{N-1;M}(N-i) \qquad (13)$$

with i = 0, 1, …, N. $K^{N;M}$ and $H^{N;M}$ are Generalized Reflection Coefficients. To get some insights into the algorithm we will describe an application to an ARMA(6,4) model. This has the Impulse Response represented in the upper half of figure 1. In the lower part we present the function $f^{N;M}(j)$, j = 0, 1, … for N = M = 4. As it can be seen, we inserted 4 gaps, but there are still non zero values for j > 4, clearly meaning that we are using too low orders.

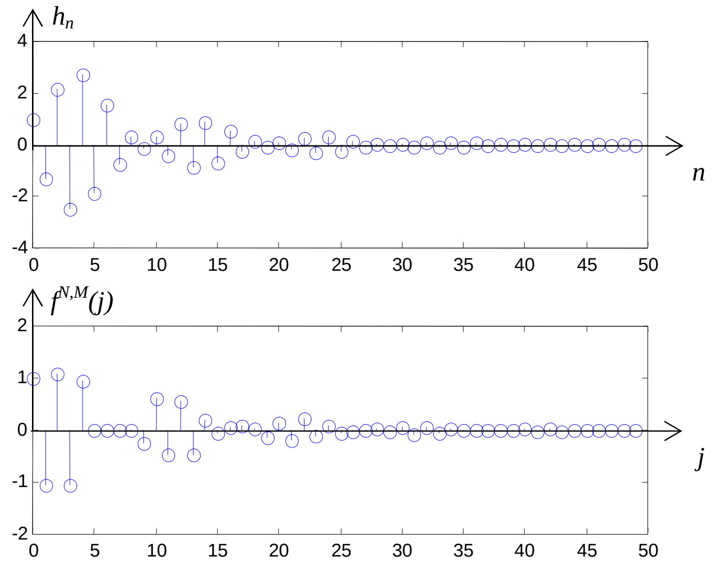

Figure 1 - Impulse Response h_n of a ARMA(6,4) system (top) and $f^{N;M}(j)$, $j = 0, 1, \ldots$ (bottom) of an ARMA(4,4) model

In figure 2, we repeat the situation but now the constructed model is an ARMA(6,4). As it can be seen, now the gaps collapsed the function for all the values above M. The nonzero values are the MA coefficients.

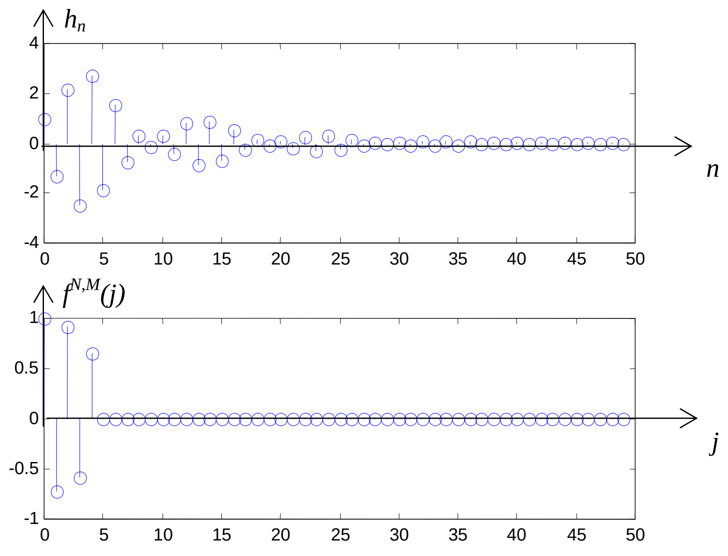

Figure 2 - Impulse Response h_n of an ARMA(6,4) system (top) and $f^{N;M}(j)$, $j = 0, 1, \ldots$ (bottom) of an ARMA(6,4) model.

The Double Levinson recursion supplies us with a very important result, which will be useful in determining the orders of the model. This result is stated in the following theorem (for proof, see [2]:)

Theorem

Let $M \geq 0$ be an integer constant and $A^{N;M}(z)$ and $\Gamma^{N;M+1}(z)$ the Z Transforms of $a^{N;M}(i)$ and $\gamma^{N;M+1}(i)$. The Nth degree polynomials $A^{N;M}(z)$ and $\Gamma^{N;M+1}(z)$ corresponding to M and M+1 zeros, respectively, are, up to a constant, reverse polynomials:

$$\Gamma^{N;M+1}(z) = \phi . z^{-N;} . A^{N;M}(z^{-1}) \qquad (14)$$

ϕ being the last coefficient of $\Gamma^{N;M+1}(z)$.

As consequence of this theorem we have:

$$H^{N;M+1}.K^{N;M}=1 \qquad (15)$$

Its proof being immediate from the theorem. This result is very interesting since it allows us to compute the correct orders:

- it is enough to run the algorithm for N,M values ranging from 0 to N_0, M_0 higher than the expected orders.
- For the correct AR order and the correct plus one MA order the product is one.

To exemplify this assertions, we will consider the following two examples.

Example 1 – AR case: consider an AR(3) with coefficients a = [1 -1.0871 1.1961 -0.4512]. In table 1 we show the $H^{N;M}.K^{N;M}$ product pattern for N, M = 0, 1, 2, 3, 4 , where "hv" means a high value obtained when the determinant of the underlying matrix is close to zero (small values of $f^{N;M}(0)$ after a product almost equal to one). This happens when the orders are oversized. From the table we conclude easily that the system is indeed an AR(3). It is interesting to remark that if we compute the poles and zero corresponding to an ARMA(4,1), the extra pole is cancelled by the zero (as it should be.)

N\M	*1*	*2*	*3*	*4*
1	*-0.012*	*hv*	*0.0082*	*-2.5803*
2	*0.6571*	*hv*	*1.1446*	*1.0401*
3	*1*	*hv*	*1*	*1*
4	*Hv*	*6.6935*	*0.2955*	*-2.1107*

Table 1 – $H^{N;M}.K^{N;M}$ product pattern for the AR(3) case

Example 2 – ARMA case: we consider now an ARMA(3,2) system defined by the previous AR parameters (example 1) and with b = [1 1.2 -1.6] as MA parameters. As before, table 2 showing the $H^{N;M}.K^{N;M}$ product pattern for N,M = 0, 1, 2 , 3, 4, suggests the correct orders. It is important to refer that:

a) Although the system is not minimum phase, we obtain the correct MA parameters;
b) If the MA order is the correct one but the AR one is oversized, there is no problem since the extra coefficients are zero.

The application of this algorithm to the pole/zero modelling of integer order continuous-time systems is also possible. To do it we only have to substitute above the Impulse Response by the sequence $h_n = \frac{(-1)^n}{n!} m_n$, n = 0, 1, … , where m_n is the sequence of the momenta of the Impulse Response of the continuous-time system given by: $m_n = \int_0^\infty h(t)t^n\, dt$.

N\M	*1*	*2*	*3*	*4*
1	*-0.0592*	*62.5179*	*0.0652*	*-1.7805*
2	*0.1984*	*0.9592*	*0.8598*	*0.9575*
3	*0.0422*	*-0.9930*	*1*	*1*
4	*0.0165*	*0*	*hv*	*0.8159*

Table 2 – $H^{N;M}.K^{N;M}$ product pattern for the ARMA(3,2) case.

The importance of (14) lies on the bridge it establishes between two different MA order polynomials. As consequence of this theorem we have, from (13) [10]:

$$A^{N-1;M}(z) = A^{N;M-1}(z) + \mu^{N;M} z^{-1}.A^{N-1;M-1}(z) \qquad (16)$$

where $\mu^{N;M}$ is obtained by forcing the Nth order coefficient to be zero:

$$\mu^{N;M} = -\frac{a^{N}_{M-1}(N)}{a^{N-1}_{M-1}(N-1)} \qquad (17)$$

The recursion (16) is very interesting since it allows us to compute the AR part of a given ARMA(N_0,M_0) from a sequence of AR models with orders ranging from 1 to $N_0 + M_0$. Using (2) we obtain

$$f^{N-1;M}(j) = f^{N;M-1}(j+1) + \mu^{N;M} f^{N-1;M-1}(j) \qquad (18)$$

that allows us to compute the MA parameters from $f^{N;0}(j)$, resulting from the Levinson recursion (12) with M = 0. In our applications we preferred to compute the MA parameters from (1), though.

We applied the recursion (16) to the model used in example 1. Immediately at the first recursion (M = 1), all the polynomial with degree greater than or equal to 4 reproduced the AR polynomial we were looking for. All the extra coefficients were zero. The same happened with the second at M = 2. When we go beyond the correct MA order the coefficient (17) becomes very high due to a division by a very small value (theoretically, zero).

III. APPLICATION TO PSEUDO-FRACTIONAL MODELLING

In this section, we assess the estimation of ARMA models for approximating discrete-time fractional models. We considered the fractional difference

$$H_{bd}(z) = (1 - z^{-1})^{\alpha}, \quad |z|>1 \qquad (19)$$

and the Tustin (bilinear)

$$H_{bil}(z)=\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right)^{\alpha} \quad |z|>1 \qquad (20)$$

as approximations to the α order differintegrator. For each one, we applied the algorithms described above and tried to find patterns that pointed us towards minima orders. We used only values of $|\alpha| < 1$, because the other cases correspond to join integer order poles and/or zeros to the ARMA model obtained with the fractional part of α.

The introduction of gaps in the function $f^{N;M}(j)$, j = 0, 1, … is presented in figure 3 for backward differences with order α= -0.4. We present the results obtained with an ARMA(6,4) and ARMA(4,6). As it can be seen, the values of the function above 4 are almost zero, meaning that, although the original system is not ARMA, it can be modelled with an ARMA. For the bilinear case, the results are similar. With this in mind we are going to study the behaviour of product $H^{N;M+1}.K^{N;M}$ through the recursion progression.

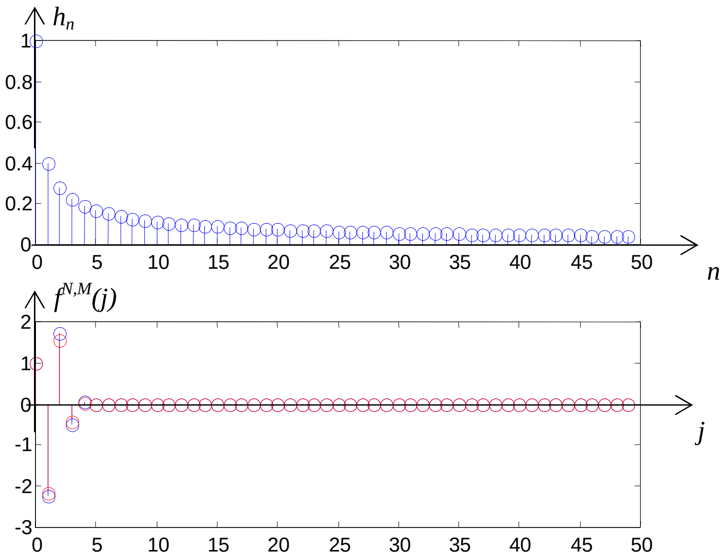

Figure 3 - Impulse Response h_n of a backward difference system (top) and $f^{N;M}(j)$, $j = 0, 1, \ldots$ (bottom) of an ARMA(4,4) and an ARMA(6,4) models.

The product $H^{N;M+1}.K^{N;M}$ pattern does not tell much but, and as in the previous tables, we find that the lower diagonal values were smaller than those in the upper diagonal in both cases (19) and (20). This suggests that the MA order is more important than the AR one. In the following tables, to better illustrate the behaviour of such patterns for the backwards difference and bilinear, all the values less than 0.9 were represented by a "0", whereas values between 0.9 and 1.1 were represented by "1" and, finally, the values above 1.1 were represented by "2".

N/M	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*
1	0	2	2	2	1	1	1	1	1	1
2	0	0	2	2	2	2	2	1	1	1
3	0	0	0	2	2	2	2	2	2	2
4	0	0	0	0	2	2	2	2	2	2
5	0	0	0	0	0	2	2	2	2	2
6	0	0	0	0	0	0	2	2	2	2
7	0	0	0	0	0	0	0	2	2	2
8	0	0	0	0	0	0	0	0	2	2
9	0	0	0	0	0	0	0	0	0	2
10	0	0	0	0	0	0	0	0	0	0

Table 3 – $H^{N;M}.K^{N;M}$ product pattern for the ARMA(10,10) corresponding to the backward difference case.

N/M	1	2	3	4	5	6	7	8	9	10
1	0	2	0	2	0	2	0	2	0	2
2	0	0	2	0	2	0	2	0	2	1
3	0	0	0	2	0	2	0	2	0	2
4	0	0	0	0	2	0	2	0	2	0
5	0	0	0	0	0	2	0	2	0	2
6	0	0	0	0	0	0	2	0	2	0
7	0	0	0	0	0	0	0	2	0	2
8	0	0	0	0	0	0	0	0	2	0
9	0	0	0	0	0	0	0	0	0	2
10	0	0	0	0	0	0	0	0	0	0

Table 4 – $H^{N;M}.K^{N;M}$ product pattern for the ARMA(10,10) corresponding to the bilinear case

To try to find other insights into the orders, we ran the recursive algorithm, allowing us to conclude that:

a) Small values of the coefficient $\mu^{N;M}$ in (17) point to correct N and M orders;

b) Very high values of $\mu^{N;M}$ mean that, at least the MA order is higher than the correct. In this situation we must decrease it. This situation corresponds to unstable model;

c) The observation of the $\mu^{N;M}$ pattern suggests an ARMA(N,N) model;

d) In the difference case, the poles and zeros of the ARMA(N,N) model are always positive and interlaced;

e) In the bilinear case, the poles and zeros of the ARMA(N,N) model are symmetric;

f) Our simulations pointed out to values of N from 7 to 10. For most situations the approximation is good in both time and frequency domains.

g) Smaller values of α results in better amplitude frequency response approximations to the exact model, the largest amplitude deviations occurring in the low-frequency regions for the difference case. For the cases where $|\alpha| > 0.5$, leading to a worst amplitude response modelling, one can always model them as a combination of two models, thus forcing the “new” fractional to be of order $(1 - \alpha)$. The additional integer term corresponds to either an integrator or a differenciator that results in an extra $(1 - z^{-1})^{-1}$ or $(1 - z^{-1})$ factor, respectively.

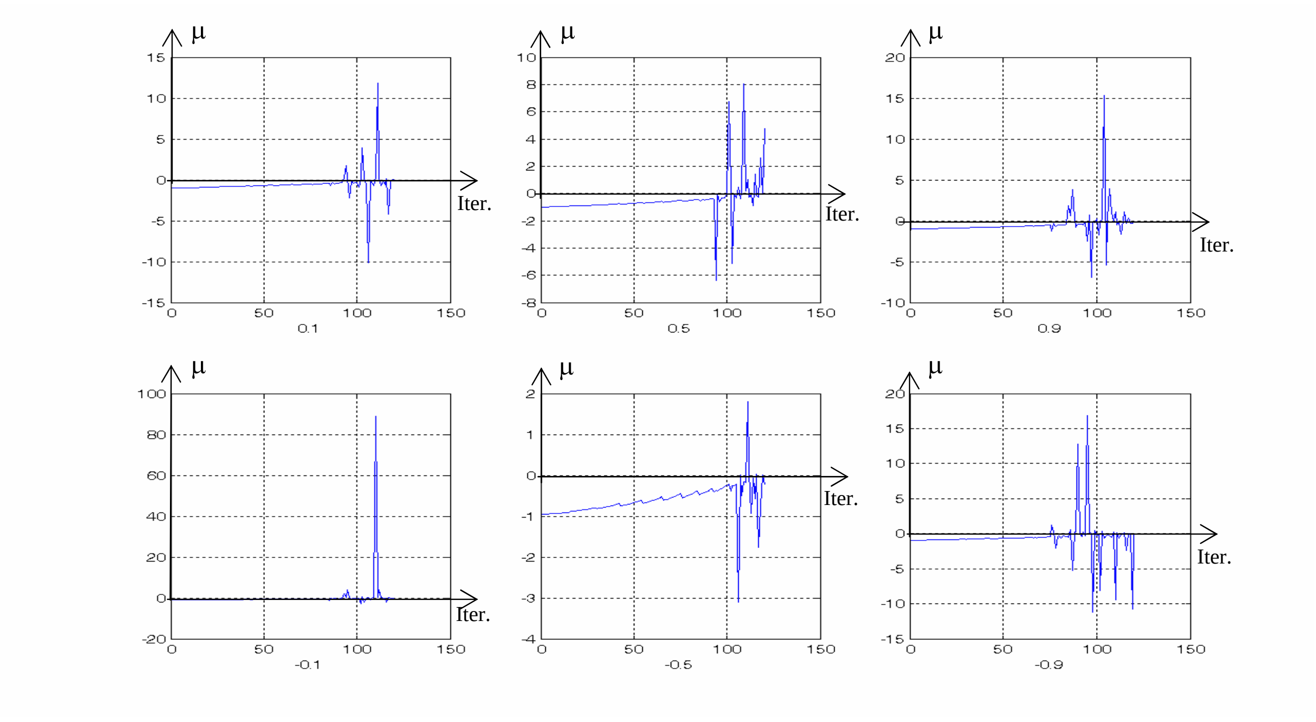

Figure 4 – the behaviour of the $\mu^{N;M}$ as a tool for orders choice, for the differentiator (above) and integrator (below), with α = 0.1 (left), α =0 .5 (middle) and α = 0.9 (right).”Iter” means model iteration.

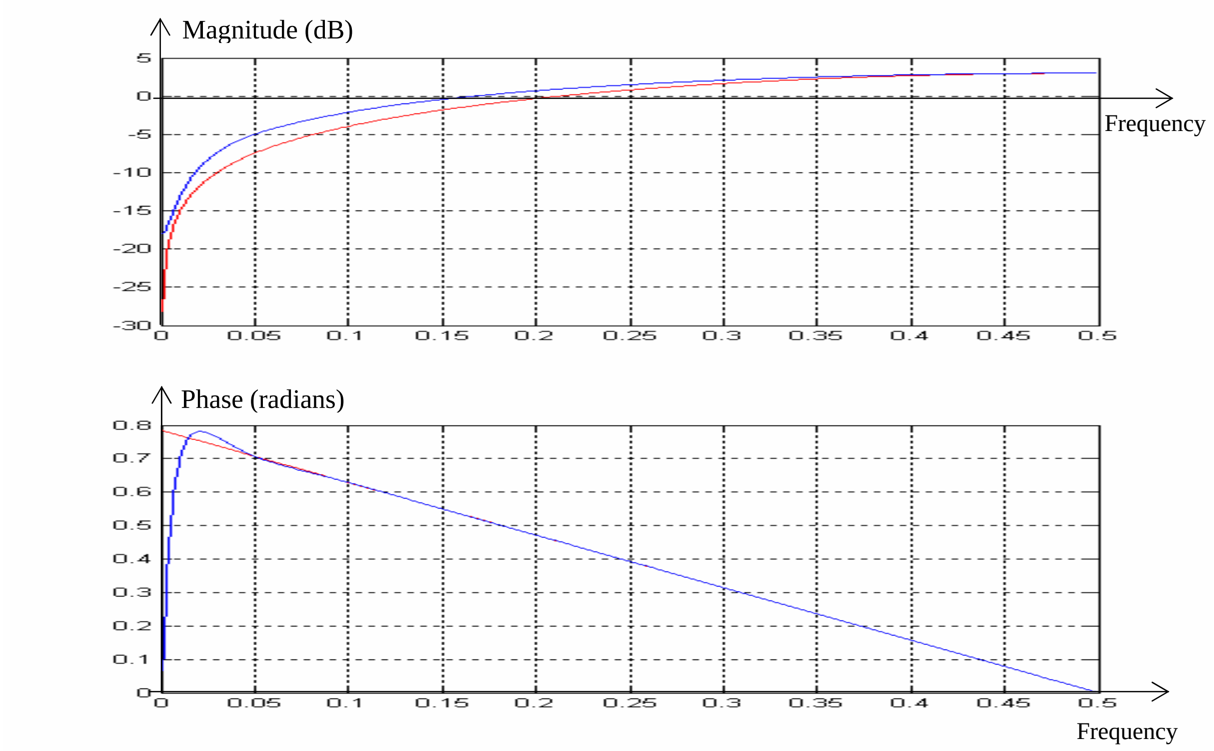

Figure 5 – ARMA(9,9) frequency response plots (in blue) for the backward difference, with α = 0.5. Upper graph is the magnitude response (in dB) and the lower graph is the phase diagram (in radian); red curves correspond to the exact model, frequency 0.5 is half the Nyquist frequency.

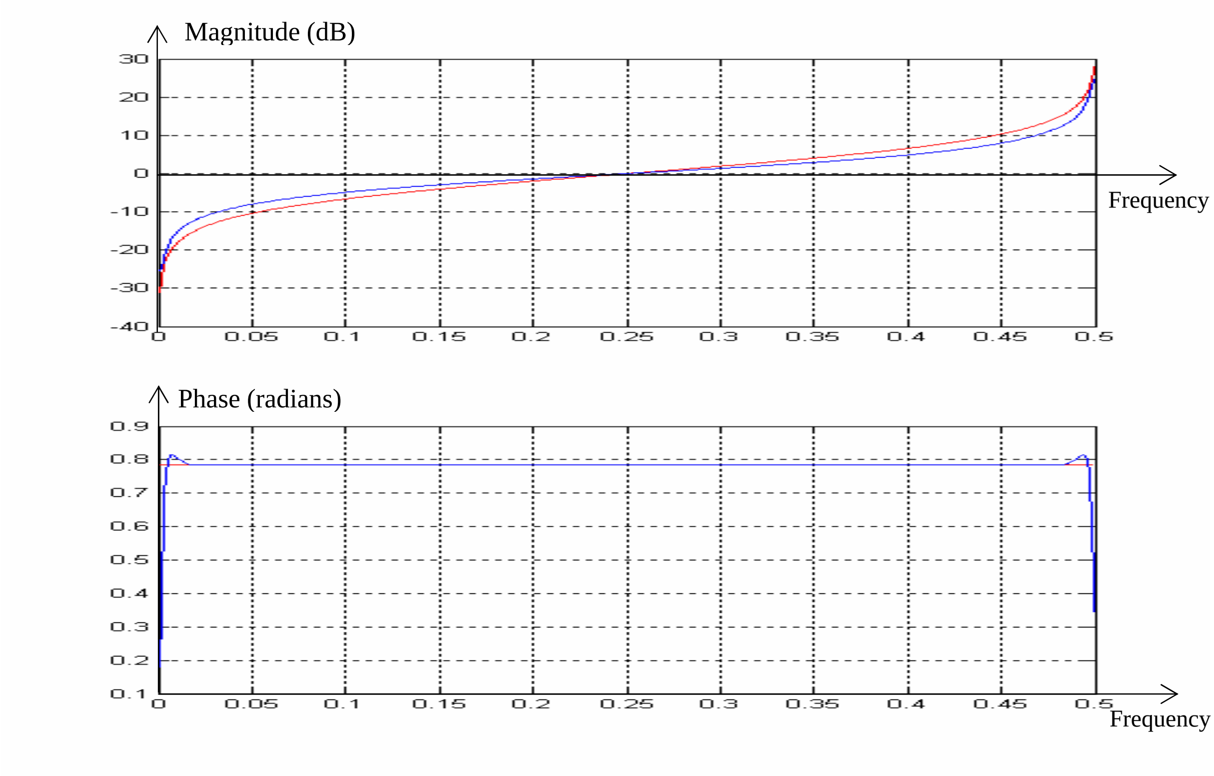

Figure 6 – ARMA(9,9) frequency response plots (in blue) for bilinear, with α = 0.5. Upper graph is the magnitude response (in dB) and the lower graph is the phase diagram (in radian); red curves correspond to the exact model, frequency 0.5 is half the Nyquist frequency.

We made also a study of the pole-zero distribution (figures 7 and 8) and found that:

a) In the backward difference case (models N,N) the poles and zeros are positive real. With higher orders complex poles and zeros may appear (above N = M= 9);

b) In the bilinear case, the poles and zeros are real. We did not find complex poles or zeros, unless N = M > 16;

c) In the bilinear case, and for some α, we obtained unstable models for some N, M < N;

d) If the pole order is the same of the zero order, the poles and zeros are interlaced. In the bilinear case, they are symmetric to each other;

e) If the orders are not equal, the "extra" poles or zeros tend to appear near zero.

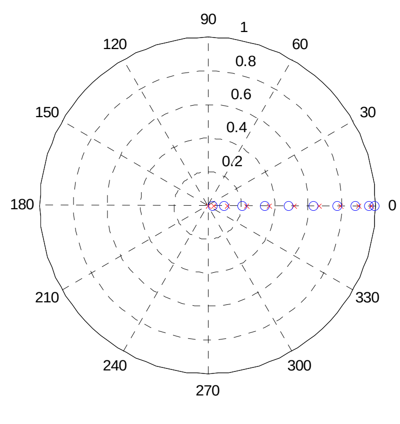

Figure 7 – ARMA(N,M) polar pole-zero plots for the backward differences with α = 0.8 for N, M = 10.

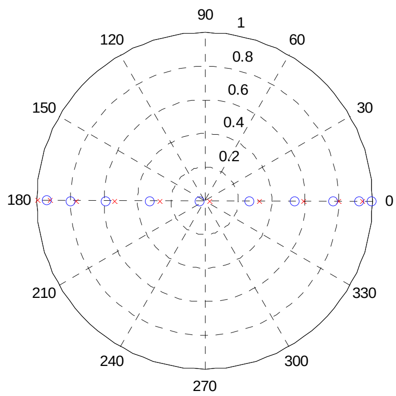

Figure 8 – ARMA(N,M) polar pole-zero plots for the bilinear with $\alpha = 0.8$ for N, M = 10.

While the last statement is more or less natural, statement d) deserves some considerations. A presence of a pole ($\alpha < 0$) or a zero ($\alpha > 0$) near z = 1 is needed, since it is the branch point of the transfer function and it has a very large influence on the function. The interlaced pole/zero pairs is needed because we know that zeros, inside the unit circle, decrease the amplitude and increase the phase, while poles increase the amplitude and decrease the phase. So the doublets pole/zero determine the variation of the amplitude and phase. We may ask why they are over the real axis. If they were not at the real axis, they would contribute to the appearing of peaks and valleys, because the amplitude and phase changed faster and the doublet effect was not so clear.

IV. CONCLUSIONS

The algorithm we have just presented gives a simple way of computing the AR and MA parameters of an ARMA model from a given Impulse Response, of a not necessarily ARMA, linear system. The algorithm is recursive and gives insights into the orders computation. We applied it to modelling two special cases of fractional linear systems: the systems with transfer functions that are fractional powers of the backward difference and bilinear transformations. We presented some examples. From them, we may conclude that we cannot give widely valid prescriptions on the modelling orders: different fractional orders imply different ARMA models. However, it seems that the pole-zero pair (doublet) has an important job in the modelling performance.

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